Product Choice and Price Discrimination in Markets with Search Frictions*

Natalia Fabra
Universidad Carlos III and CEPR

Juan-Pablo Montero
PUC-Chile

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Abstract

In a seminal paper, Champsaur and Rochet (1989) showed that competing firms choose non-overlapping qualities so as to soften price competition at the cost of giving up profitable opportunities to price discriminate. In this paper we show that an arbitrarily small amount of search frictions is enough to give rise to an equilibrium with overlapping qualities. This is in contrast to other sources of market power (e.g. horizontal product differentiation), which have to be sufficiently strong in order to give rise to overlapping qualities. In markets with search frictions, competing firms face the monopolist's incentive to price discriminate, which induces them to offer the full product line even if this forces them to compete head-to-head. Hence, search frictions increase prices and reduce consumers surplus for given product choices, but they can also lead to lower prices and higher consumer surplus whenever they induce firms to offer broader and overlapping quality offerings.

Keywords: second degree price discrimination, search, vertical differentiation, retail competition.

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*Emails: natalia.fabra@uc3m.es and jmontero@uc.cl. We are indebted to Mar Reguant for her contribution to this project. Dolores Segura and Francisco Cabezón provided excellent research assistance. Heski Bar-Isaac, Guillermo Caruana, Daniel García, Tom Gresik, Michele Polo, Wojciech Olszewski and seminar audiences at CEMFI, Northwestern, Notre Dame, the University of Chicago, the University of Pennsylvania, PUC-Rio, the University of Arizona, the Aalto University (Helsinki) and EARIE 2017 (Maastrich) provided valuable comments. Fabra is grateful to the Economics Department of Northwestern University for their hospitality while working on this paper, and the Spanish Ministry of Education (Grant ECO2016-78632-P) for financial support. Montero is grateful to the Complex Engineering Systems Institute - ISCI (CONICYT-PIA-FB0816) and Fondecyt (Grant No. 1161802) for financial support.

1 Introduction

Since the classical work of Chamberlin (1933), a well known principle in economics is that firms differentiate their products in order to relax competition. Champsaur and Rochet (1989) (CR, thereafter) formalized this Chamberlinian incentive in a model in which quality choices are followed by price competition.[1] They showed that firms choose non-overlapping qualities because the incentives to soften price competition dominate over the incentives to better discriminate heterogeneous consumers. Yet, in many markets, competing firms often carry overlapping qualities even when this creates fierce competition among them.

In this paper, we show that in the presence of search frictions CR's prediction need not hold. When consumers are not perfectly informed about firms' prices and qualities, they cannot choose their preferred option unless they incur search costs to learn and compare all options. Since the seminal work of Diamond (1971), the search literature has shown that the introduction of search frictions can have substantial effects on competition, no matter how search is modeled.[2] However, unlike CR, this literature has broadly neglected the possibility that firms engage in price discrimination through quality choice.[3] The main goal of this paper is to understand the interaction between search frictions and price discrimination, and their effects on product choice and pricing by competing firms.

By introducing search costs *à la* Varian (1980) in a simplified version of CR's model, we show that an arbitrarily small amount of search frictions is all it takes for firms to offer overlapping qualities in equilibrium. More generally, this equilibrium always exists as long as search frictions are present (regardless of whether they are large or small). In this sense, the equilibrium with overlapping qualities is particularly robust. In contrast, CR's equilibrium with non-overlapping qualities breaks down if the costs of providing high quality are high enough or if search frictions are substantial. The reason is that, when search is costly, the marginal incentives faced by firms mimic those of a monopolist, even if profits (which do depend on search frictions) are below the monopoly level. In other words, firms' incentives to discriminate heterogeneous consumers through quality

[1]Shaked and Sutton (1982) formalized the same idea in a model similar to Champsaur and Rochet (1989)'s, with the difference that firms are allowed to offer one quality only. Thus, in Shaked and Sutton (1982), there is no possibility to discriminate consumers at the firm level.

[2]Search models can essentially be classified as models of either simultaneous search (Burdett and Judd, 1983) or sequential search (Stahl, 1989). De los Santos *et al.* (2012) test which of the two processes best represents actual search for online books, and conclude in favor of the simultaneous search model, which is the approach we adopt in this paper.

[3]Unlike the current paper, in which we model second-degree price discrimination, Fabra and Reguant (2017) allow for third-degree price discrimination in markets with search costs.

choices dominate over the incentives to soften price competition, thus inducing firms to offer the full quality range.

We show that the comparative statics of equilibrium prices and consumer surplus with respect to search frictions can be biased if quality choices are taken as given. Essentially, search frictions affect quality choice, and through that, they end up affecting prices and consumer surplus. There are two effects at play: on the one hand, an arbitrarily small amount of search frictions intensifies competition by giving rise to overlapping quality choices; on the other, further increases in search frictions relax competition, eventually leading to prices above those in frictionless markets. In sum, while an increase in search frictions is in general anti-competitive, search frictions might also lead to lower prices and higher consumer surplus when they induce firms to carry broader and overlapping quality choices.[4]

Beyond investigating the effects of search frictions on firms' quality choices, we also aim at understanding their effects on equilibrium pricing in general. We show that the incentive compatibility constraints faced by multi-product firms introduce an important departure from Varian (1980): the prices for the various goods sold within a store cannot be chosen independently from each other. This has several implications for pricing behavior. For instance, in the case in which both firms carry the two goods, if competition becomes particularly intense, firms reduce the relative price of the high versus the low quality good below the level that is necessary to induce separation of consumers' types. In other words, during periods of sales *à la* Varian, the incentives to compete may dominate over the incentives to minimize information rents. Yet, even though competition reduces prices and relative prices, the relative mark-ups remain unchanged. Additionally, incentive compatibility considerations imply that multi-product firms tend to charge lower prices on average as compared to single-product firms, contrary to the analysis of pricing by single-product versus multi-product monopolists under complete information.

Our paper is related to two strands of the literature: (*i*) papers that analyze competition with search costs, and (*ii*) papers that characterize quality choices under imperfect competition.[5] The vast part of the search literature assumes that consumers search for

[4]In general, search costs are thought to relax competition, thus leading to higher prices, although not as intensively as the Diamond paradox would have anticipated (Diamond, 1971). There are some exceptions to this general prediction. Some recent papers have shown that search costs can lead to lower prices, particularly so when search costs affect the types of consumers who search. For instance, see Moraga-González *et al.* (2017) and Fabra and Reguant (2017).

[5]There is also a large empirical literature investigating price discrimination in markets where search costs matter, with a focus on price patterns. There are studies on gasoline markets, where consumers have the choice of paying for full-service or self-service gasoline at the same station, or of searching for competing stations (Shepard, 1991); the airline industry, where travellers can choose whether to fly in

one unit of an homogenous good, with two exceptions. Some search models allow for product differentiation across firms but, unlike ours, assume that each firm carries a single product.[6] Other search models allow firms to carry several products but, unlike ours, typically assume that consumers search for more than one ('multi-product search').[7] In these models, consumers differ in their preference for buying all goods in the same store ('one-stop shopping') rather than on their preferences for quality.[8] These differences are relevant. In the first type of search models, the single-product firm assumption leaves no scope for price discrimination within the firm. Hence, pricing is solely driven by competitive forces. In the second type of search models, the multi-product search assumption implies that discrimination is based on heterogeneity in consumers' shopping costs, which become the main determinant of firms' product choices (Klemperer, 1992).

Within the 'multi-product search' literature, two papers deserve special attention. In line with our results, Zhou (2014) finds that multi-product firms tend to charge lower prices than single-product firms. This is not driven by the interaction between competition and price discrimination, as in our paper, but rather by a 'joint search' effect, i.e., multi-product firms charge less because they gain more by discouraging consumers from searching competitors. In Rhodes and Zhou (2016), increases in search costs imply that consumers value one-stop shopping more, thus making it more likely that the equilibrium involves multi-product firms. Unlike us, for small search costs, Rhodes and Zhou (2016) predict asymmetric market structures with single-product and multi-product firms coexisting. The driving force underlying our predictions is quite different: since in our model consumers buy a single good, the multi-product firm equilibrium is not driven by one-stop shopping considerations but rather by firms' incentives to price discriminate consumers

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business or in economy class, or just in economy class but with certain restrictions (Borenstein and Rose, 1994; Gerardi and Shapiro, 2009); coffee shops (McManus, 2000), cereals (Nevo and Wolfram, 2002), theaters (Leslie, 2004), Yellow Pages advertising (Busse and Rysman, 2005), and cable TV (Crawford and Shum, 2007), among others.

[6]For models with horizontal product differentiation, see for instance Anderson and Renault (1999) and Bar-Isaac *et al.* (2012); see Ershow (2017) for an empirical application. Wildenbeest (2011) allows for vertical differentiated products but, unlike us, assumes that all consumers have the same preference for quality; hence, there is no scope for price discrimination. He finds that all firms use the same symmetric mixed strategy in utility space, which means that firms use asymmetric price distributions depending on the quality of their product. In contrast, we find that firms might use different pricing strategies for the same product, with this asymmetry arising because of price discrimination within the store.

[7]There is a recent strand of papers in the ordered search literature that analyze obfuscation by multi-product firms (Gamp, 2016; Petrikaite, 2017). Their emphasis is on the monopoly case. See Armstrong (2016) for a discussion.

[8]One-stop shopping considerations are also the driving force behind the evidence of price dispersion across stores documented by Kaplan *et al.* (2016).

with heterogenous quality preferences. Despite these differences, our paper has one common prediction with both Rhodes (2014) and Rhodes and Zhou (2016): namely, search frictions can give rise to lower prices through their effect on endogenous product choices.

As far as we are aware of, Garret *et al.* (2016) is the only paper that, like ours, introduces frictions in a model of price competition in which firms can carry more than one product but in which consumers buy only one.[9] There are however two important distinctions between the two analysis. First, in Garret *et al.* (2016), firms decide qualities and prices *simultaneously*, rather than sequentially. These two approaches apply to different settings and help shed light on different issues. The simultaneous timing is appropriate in settings where firms can change product design rather quickly, or alternatively, when firms commit to prices for long periods of time; for example, under long term contracts. As such, it allows to analyze the degree of substitution between price and quality in firms' strategies. In contrast, the sequential timing is better suited to capture the notion that in many markets firms can change prices at will, while changes in product line decisions occur less often as these usually involve changes in the production and/or retail facilities (Brander and Eaton, 1984). This distinction is relevant as in simultaneous settings firms cannot affect competition by pre-committing to quality, which is a fundamental driving force of our results. Furthermore, Garret *et al.* (2016) focus on symmetric equilibria which necessarily involve overlapping product lines, and hence do not explore whether equilibria with non-overlapping product choices could arise in a simultaneous choice setting.

Still, given that in our set-up firms are endogenously symmetric, our analysis shares some common predictions with Garret *et al.*'s concerning the comparative statics of prices and relative prices. Like them, in symmetric settings we find that the relative price of the two goods goes down when competition is particularly intense (in our model, when firms price both goods at the lower bound of the price supports). However, this prediction does not always extend to asymmetric product configurations, in which relative prices remain constant both at the upper as well as at the lower bound of the price supports.

Last, our paper also relates to the literature that analyzes quality choices followed by imperfect competition, either quantity competition (Gal-Or, 1983; Wernerfelt, 1986; Johnson and Myatt 2003, 2006 and 2015) or price competition with horizontal differen-

[9] Another set of related papers analyze pricing for add-ons. Ellison (2005) and Verboven (1999) consider models in which consumers are well informed about base product prices but don't know the price of the add-ons, unless they search. Critically, in these models the customers that are more likely to buy the add-ons are also less likely to search. Our model is not a model of add-on pricing because shoppers observe all prices and non-shoppers only those of the store they visit, and this applies symmetrically for both products regardless of their quality. Furthermore, our results hold regardless of whether there is correlation or not between consumers' quality preferences and search cost types.

tiation (Gilbert and Matutes, 1993; Stole, 1995). As already noted by CR (p. 535), one of the main consequences of less competitive pricing is to induce wider and, very likely, overlapping product lines. While one may view search frictions as equivalent to other forms of imperfect competition, they are not. In models of imperfect competition, for the equilibrium with overlapping (i.e., symmetric) quality choices to exist, competition has to be sufficiently weak, e.g. as shown by Gal-Or (1983), under Cournot competition, the number of firms has to be sufficiently small. The same insight also applies to models of price competition with horizontal product differentiation. If there is little (horizontal) product differentiation, the equilibrium with overlapping product choices breaks down because the rents lost when dropping a low quality good are small as compared to the increase in profits from softening competition (Wernerfelt, 1986). In contrast, the impacts of search frictions on product choices are different. Even if search frictions are arbitrarily small, firms do not have incentives to deviate from the equilibrium with overlapping product choices. The reason is that search frictions restore firms' monopoly power over those consumers who do not search (the *non-shoppers*), even when competition for those who search (the *shoppers*) is very fierce. This conclusion remains valid regardless of whether the *non-shoppers* visit one store at random, or whether they visit the one that gives them higher ex-ante utility.[10]

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 shows that in the absence of search frictions firms escape the Bertrand paradox by carrying non-overlapping product lines. In contrast, Section 4 shows that an arbitrarily small amount of search frictions is enough to induce firms to choose overlapping product lines even if this drives prices close to marginal costs. Section 5 characterizes equilibrium pricing for all potential product choice configurations, as well as the Subgame Perfect Equilibrium product choices for all levels of search frictions. Section 6 discusses the robustness of the model to several extensions. Section 7 concludes, and proofs are postponed to the Appendix.

[10]Indeed, we show that directed search by the non-shoppers strengthens our results as firms have a stronger reason to become multi-product as compared to the case when product choices are non-observable and non-shoppers search randomly. See Section 6.

2 The Model

2.1 Model Description[11]

Consider a market served by two competing firms (which we sometimes refer to as *stores*), which carry either one or two goods: either a good with high quality q^H and high costs c^H, or another one with lower quality q^L and lower costs c^L, or both.[12] We use $\Delta q \equiv q^H - q^L > 0$ and $\Delta c \equiv c^H - c^L > 0$ to denote the quality and cost differences across goods.[13]

There is a unit mass of consumers who buy at most one good. Consumers differ in their preferences over quality. A fraction λ have a low valuation for quality θ^L, while the remaining $1-\lambda$ fraction of consumers have a high quality valuation θ^H, with $\Delta\theta \equiv \theta^H - \theta^L > 0$.[14] As in Mussa and Rosen (1978), a consumer of type $i = L, H$ who purchases good $j = L, H$ at price p^j obtains net utility $u^i = \theta^i q^j - p^j$. We assume that the gross utility of a low type (high type) from consuming the low (high) quality product always exceeds the costs of producing it, i.e., $c^i < \theta^i q^i$ for $i = L, H$.

The timing of the game is as follows. First, firms simultaneously decide which product(s) they offer for sale (or "product line"). Once chosen, firms observe the product line of the rival but consumers don't. Second, firms simultaneously choose the prices for the product(s) they carry and consumers visit the stores in order to learn firms' product choices and their respective prices. We will write (ϕ_i, ϕ_j) to denote firms' product choices, with $\phi_i \in \{\varnothing, L, H, LH\}$, and use $\Pi(\phi_i, \phi_j)$ to denote the profits of firm i at the pricing stage given those product choices, $i \neq j \in \{1, 2\}$.

Following Varian (1980), we assume that there is a fraction $\mu \leq 1$ of consumers who always visit the two stores (the *shoppers*), and hence know where to find the cheapest product of each quality type.[15] Since the remaining $1-\mu$ fraction of consumers only visit

[11] To fix ideas, one can think of a market that fits well our modeling framework: the market for online books. Online stores compete to sell books of exogenously given qualities (e.g. hardcover versus paperback of a given title) to consumers with heterogenous preferences for quality. Our discussion paper, Fabra and Montero (2017), explores this analogy in more detail.

[12] Without substantial effort, our model could be interpreted as one of quantity discounts, with firms offering the different quantities of the same product to consumers with either low or high demands. Results would go through as long as costs are not linear in the quality; for instance, if bigger bundles require costly product design features, such as packaging.

[13] We can think of these costs as the wholesale prices at which retailers buy the products from either competitive manufacturers, or from a monopoly manufacturer. Endogenizing the qualities of the products or the costs faced by the retailers is out of the scope of this paper.

[14] It is possible to extend our main results to the case in which there are N potential qualities, N consumer types and N competing firms, as discussed in Section 6.

[15] There is a fundamental distinction between introducing search costs which are equal across con-

one store (the *non-shoppers*),[16] they can compare the prices of the goods sold *within* the store they have visited, but not *across* stores. We assume that the non-shoppers visit one of the two stores with equal probability.[17] Once consumers have visited the store(s), they buy the product that gives them higher utility, provided it is non-negative. In case of indifference, low (high) type consumers buy the low (high) quality product. In what follows, we will use the fraction of non-shoppers $1-\mu$ as a proxy for search frictions. Accordingly, the higher μ the lower the search frictions, with $\mu = 1$ representing a frictionless market.

2.2 Preliminaries

We start by characterizing the benchmark solutions of monopoly and marginal-cost pricing. This will serve to introduce some concepts and assumptions to be used in the rest of the analysis.

The monopoly solution A monopolist carrying both products that is able to perfectly discriminate consumer types would extract all their surplus by charging the (unconstrained) monopoly prices $p^i = \theta^i q^i$, for a per unit profit of $\pi^i = \theta^i q^i - c^i$, $i \in \{L, H\}$. This holds true regardless of the number of consumers of each type, and regardless of the relative profitability of serving one type or another. This is no longer true as we move to the more relevant case of a multi-product monopolist that cannot perfectly discriminate across consumers.

At the optimal solution, the following incentive compatibility constraints must hold

$$\theta^i q^i - p^i \geq \theta^i q^j - p^j, \tag{IC^i}$$

for $i, j \in \{L, H\}$ and $i \neq j$, which can also be re-written as

$$p^i \leq \theta^i q^i - \left(\theta^i q^j - p^j\right).$$

sumers, versus introducing a fraction of uninformed consumers. It is well known that in a (single-product, homogeneous good) Bertrand model, the former gives rise to the Diamond Paradox, such that all firms charge the monopoly price and consumers do not engage in search. A similar outcome would arise in our set-up. Varian's approach, which we adopt here, avoids the Diamond Paradox. Furthermore, as already noted, the empirical evidence reports that a large fraction of consumers are uninformed (De los Santos *et al.*, 2012).

[16]An implicit assumption is that the fractions μ and λ are uncorrelated. As we discuss in Section 7, our main results do not change if we allow for correlation between μ and λ.

[17]In some settings it may be reasonable to assume that non-shoppers observe product lines but not their prices. Accordingly, we have also considered the case in which non-shoppers visit the store that gives them higher expected utility (and split randomly between the two stores in case of symmetry). The main results of the paper are strengthened. See Section 6.

The second term on the right-hand side of the inequality represents consumers' *information rents*, i.e., the minimum surplus a type i consumer needs to obtain to be willing to buy good i instead of good $j \neq i \in \{L, H\}$. This expression already highlights an important trade-off between competition and firms' incentives to discriminate through quality choices. In particular, firms find it relatively less appealing to carry good $j \in \{L, H\}$ when the other firm is also carrying it, not only because competition reduces profits, but also because a lower p^j increases the information rents on good $i \neq j \in \{L, H\}$. This trade-off will play an important role in the analysis that follows.

Our first assumption is standard in models of second-degree price discrimination by a monopolist (Tirole, 1988): a monopolist carrying both goods finds it optimal to sort consumers out. For the multi-product monopolist, the incentive compatible (i.e., constrained monopoly) prices are thus

$$\begin{aligned} p^L &= \theta^L q^L \text{ and} \\ p^H &= \theta^H q^H - \Delta\theta q^L \\ &= \theta^L q^L + \theta^H \Delta q. \end{aligned}$$

The alternative for the monopolist is to only sell good H to the high types at the (unconstrained) monopoly price $\theta^H q^H$, thus avoiding to leave information rents to the high types but also giving up the profits on good L.[18] To guarantee that this alternative is indeed less profitable than selling the two goods requires that the profit from selling good L to the low types be enough to compensate for the information rents that must be left with the high types:[19]

$(A1)$ $\lambda \pi^L \geq (1-\lambda)\Delta\theta q^L$.

Note that $(A1)$ is evaluated at monopoly prices. Assuming that a monopolist prefers to carry all qualities does not necessarily imply that the same holds true when competition drives prices below their monopoly level.

The competitive solution Our second assumption is also standard in models of second-degree price discrimination by competing firms: there is no 'bunching' at the competitive solution. This requires marginal-cost pricing to be incentive compatible,

[18]Note that this alternative assumes that serving the high types with product H is more profitable than serving all consumers with product H at price $\theta^L q^H$. This is guaranteed by our assumption $(A3)$ below.

[19]In turn, $(A1)$ also guarantees that the optimal price for a monopolist that only carries good L is $\theta^L q^L$. In particular, serving all consumers at this price is more profitable than just serving the high types at $\theta^H q^L$.

which is equivalent to assuming that the high types are willing to pay for the extra cost of high quality whereas the low types are not:

$(A2)$ $\Delta c \in \left(\theta^L \Delta q, \theta^H \Delta q\right)$.

Implicit in $(A2)$ is the standard property that the cost of providing quality must be strictly convex in quality, i.e., $c^H/q^H > c^L/q^L$; otherwise, either type would buy the high quality product or nothing at all (Champsaur and Rochet (1989) and Johnson and Myatt (2006) adopt a similar assumption).[20]

Last, in order to reduce the number of cases we need to consider without affecting our results, we assume $\lambda \geq 1/2$ and:

$(A3)$ $(1-\lambda)(\pi^H - \Delta\theta q^L) \geq \theta^L q^H - c^H$.

Assumption $(A3)$ simply states that a firm prefers to sell product H to the high types at the constrained monopoly price, rather than reducing it to $\theta^L q^H$ so as to also serve the low types.[21]

Minmax profits Inspection of assumption $(A2)$ above allows to obtain useful expressions for the analysis of the model. As implied in $(A2)$, the maximum profits that can be made out of product $i \in \{L, H\}$ when good $j \neq i$ is priced at marginal costs are strictly positive. Since firms would never sell their products below marginal costs, these constitute *minmax profits*. In particular, if good L is sold at c^L, good H can at most be sold at the highest price that satisfies the high types' incentive compatibility constraint, i.e., $p^H \leq c^L + \theta^H \Delta q$, thus giving per unit profits of

$$\varphi^H \equiv \theta^H \Delta q - \Delta c > 0.$$

The minmax profits for good H are always strictly below monopoly profits π^H given that, for all values of c^L, good L imposes a competitive constraint on good H.

In turn, if good H is sold at c^H, good L can at most be sold at the highest price that satisfies the low types' participation and incentive compatibility constraints, i.e., $p^L \leq \min\left\{\theta^L q^L, c^H - \theta^L \Delta q\right\}$, thus giving per unit profits of

$$\varphi^L \equiv \min\left\{\pi^L, \Delta c - \theta^L \Delta q\right\} > 0.$$

For $c^H \geq \theta^L q^H$, the participation constraint binds first, so that good L can be sold at the monopoly price even when good H is priced at marginal cost. Alternatively, for

[20] Note also that convexity ensures that there is a non-empty region of λ values for which $(A1)$ and $(A2)$ are valid.

[21] Note that assuming $c^H \geq \theta^L q^H$ would make $(A3)$ redundant as the left hand side is always positive.

$c^H < \theta^L q^H$, the incentive compatibility constraint binds first, so that the minmax profits for good L are strictly below monopoly profits.

In sum, the per unit profits that a firm that monopolizes good $i \in \{L, H\}$ loses when product $j \neq i$ is made available at marginal cost equal $\pi^i - \varphi^i \geq 0$, with equality only for good L when $c^H \geq \theta^L q^H$.[22]

We are now ready to solve the game. We start by analyzing the case in which all consumers are shoppers, $\mu = 1$ (i.e., no search frictions), then move on to introducing an arbitrarily small fraction of non-shoppers, $\mu \to 1$, and finish by providing a full equilibrium characterization for all parameter values, $\mu \in [0, 1)$.

3 Escaping the Bertrand Paradox

In this section we characterize the Subgame Perfect Equilibrium (SPE) in the absence of search frictions, i.e., under the assumption that all consumers are shoppers, $\mu = 1$. Following CR, we focus on equilibria at which both firms make a strictly positive profit.

Our first result is fully in line with CR, who were the first to show that simultaneous quality choices followed by price competition give rise to non-overlapping product lines.

Proposition 1 *Assume $\mu = 1$. The "specialization equilibrium" (L, H) is the unique Subgame Perfect Equilibrium (SPE).*[23] *Equilibrium prices are strictly above marginal costs.*

Proof. See the Appendix. ■

Under product choices (L, H), there does not exist a pure strategy price equilibrium. This stems from an important result: in equilibrium, firms' prices must satisfy incentive compatibility. Otherwise, the firm selling good H would sell nothing and would thus be better off reducing its price to satisfy incentive compatibility. However, if the high types' incentive compatibility constraint is binding, the firm carrying good L could in turn attract all customers by slightly reducing its own price. Since these opposing forces destroy any candidate in pure strategies, the equilibrium has to be in mixed strategies. Furthermore, all prices in the support of the mixed strategies are strictly above marginal costs.

[22] We will sometimes express profit expressions as functions of φ^H and φ^L. The following equalities will be particularly useful throughout the analysis: $\pi^H - \Delta\theta q^L = \pi^L + \varphi^H$, and if $c^H < \theta^L q^H$ then $\theta^L q^H - c^H = \pi^L - \varphi^L$.

[23] If $c^H < \theta^L q^H$, there also exists a symmetric mixed strategy equilibrium with positive profits such that firms choose L and H with positive probability. On the contrary, if $c^H \geq \theta^L q^H$, this equilibrium does not exist as it is dominated by playing LH.

This has meaningful implications for equilibrium product choices. First, since at (L,H) product L is priced *above* marginal costs, profits on good H are strictly above its minmax. If firm H deviated to also carrying good L, p^L would be driven down to marginal costs. Hence, the profits on good L would be zero and the profits on good H would be driven down to its minmax, making such a deviation unprofitable. Similarly, since at (L,H) product H is priced *above* marginal costs, profits on good L are (weakly) above its minmax. If firm L deviated to also carrying good H, it would make no profits on good H and would (weakly) reduce its profits on good L as competition for good H becomes fiercer.[24] In sum, since neither firm can gain by deviating from (L,H), the "specialization equilibrium" constitutes a SPE of the game with no search frictions. Since at any other product choice configuration at least one firm would make zero profits, this is the unique equilibrium of the game that satisfies CR's focus on equilibria with strictly positive profits for both firms.

In the next section we show that this prediction is not robust to introducing search frictions.

4 Back to the Bertrand Paradox

Before solving the game for all $\mu \in [0,1)$, in this section we show that an arbitrarily small amount of search frictions $\mu \to 1$ is enough to give rise to an equilibrium with overlapping product lines. Furthermore, we show that if the costs of providing high quality are high enough, the "specialization equilibrium" no longer exists.

To explore this in more detail, let us first analyze pricing incentives at the subgame with product choices (LH,LH) ("overlapping"). Search frictions, no matter how small, imply that marginal cost pricing is not in equilibrium as firms could make positive profits out of the non-shoppers. Similarly, setting prices at the (constrained) monopoly level is not in equilibrium either as firms would have incentives to charge slightly lower prices so as to attract the shoppers. More generally, search frictions rule out any equilibrium candidate in pure strategies as firms face a trade off between charging high prices to exploit the non-shoppers versus charging low prices to attract the shoppers. Since firms must be indifferent between charging any price in the support, expected equilibrium profits can be computed by characterizing profits at the upper bound, where firms optimally serve

[24] If $c^H \geq \theta^L q^H$, the firm carrying good L makes the same profits at (L,H) as at (LH,H) since good H does no impose a competitive constraint on good H. In any event, firm L could increase its profits to also carrying good H.

their share of non-shoppers at (constrained) monopoly prices,

$$\Pi(LH, LH) = \frac{1-\mu}{2} \left[\lambda\pi^L + (1-\lambda)(\pi^H - \Delta\theta q^L)\right]. \tag{1}$$

Importantly, each firm's equilibrium profits are a fraction $(1-\mu)/2$ of the multi-product monopolist's profits because firms only make profits out of the non-shoppers. This is true in expectation only, as for prices below the upper bound firms make profits out of the shoppers too (which compensate for the lower profits made out of the non-shoppers at prices below the upper bound). As μ approaches 1 and all customers become shoppers, the equilibrium price distributions concentrate around marginal costs, and firms' profits are driven down to (almost) zero. The Bertrand outcome is thus restored.

Could firms escape from the Bertrand paradox by having one of them drop one product, either L or H?[25] Let us first analyze the incentives of moving from (LH, LH) to (H, LH). Since a pure strategy equilibrium does not exist, and firms have to be indifferent across all prices in the support, expected profits for product H equal those of serving the non-shoppers at the upper bound. Since firm H is not constrained by incentive compatibility, its optimal price at the upper bound is the (unconstrained) monopoly price. Its expected profits become

$$\Pi(H, LH) = \frac{1-\mu}{2}(1-\lambda)\pi^H. \tag{2}$$

Since firm H's profits are a fraction $(1-\mu)/2$ of monopoly profits, comparing (1) and (2) is equivalent to assessing the monopolist's incentives to carry the high quality good only versus carrying both goods. Assumption $(A1)$ guarantees that (1) exceeds (2) as the losses from not selling the low quality product exceed the information rents left to the high types. Thus, even though product L erodes the rents made on product H, the firm is better off carrying it.

The alternative is for one of the two firms to drop product H, thus moving from (LH, LH) to (L, LH). Now, the expected profits of firm L must be equal to the profits of serving all the non-shoppers at the unconstrained monopoly price,[26]

$$\Pi(L, LH) = \frac{1-\mu}{2}\pi^L,$$

again a fraction $(1-\mu)/2$ of monopoly profits. This payoff is strictly less than (1) since the firm gives up the extra profit that firm L could make by selling the high quality good to the non-shopper high types.

[25] No firm has incentives to drop both products altogether as they both make positive profits at (LH, LH).

[26] Note that in this case the firm would serve both the low and the high-types, since the latter are also willing to buy the low quality product at the unconstrained monopoly price for product L.

In sum, firms' profits are the same *as if* they exploited their monopoly power over the non-shoppers and competed fiercely for the shoppers, obtaining no profits out of the latter. Hence, firms' incentives to price discriminate through product choice mimic those of the monopolist. Consequently, in the presence of infinitesimally small search frictions, there exists a SPE with overlapping product lines (LH, LH), in stark contrast with CR's prediction.

To assess whether this equilibrium is unique or not, let us first note that the "specialization" equilibrium of Proposition 1 is ruled out when the cost of providing high quality is sufficiently large, $c^H \geq \theta^L q^H$ (or equivalently, when the costs of providing quality is sufficiently convex). Starting at (L, H), firm L is strictly better off adding product H given that under (LH, H) it can now price discriminate the non-shoppers without eroding its profits on good L. Indeed, the firm would be able to increase its profits by $(1-\mu)(1-\lambda)\varphi^H/2 > 0$ from selling the high rather than the low quality product to the non-shopper high types, while it would still make profits $\lambda\pi^L$ out of the low types.

In contrast, if the costs of high quality are sufficiently low, the addition of good H erodes the rents of good L, making firm L worse off: the rents on product H are infinitesimally small while the profits on good L would go down by $\lambda\left(\pi^L - \varphi^L\right) > 0$. Similarly, firm H does not want to add product L as its profits would fall by $(1-\lambda)\left(\pi^H - \varphi^H\right) > 0$. Thus, the "specialization" equilibrium survives the introduction of infinitesimally small search frictions but *only* when the costs of providing high quality are sufficiently low.

Our second proposition summarizes these results.

Proposition 2 *Assume* $\mu \to 1$.

(i) The "overlapping" equilibrium (LH, LH) *constitutes a SPE. Equilibrium prices approximate marginal costs.*

(ii) The "specialization" equilibrium (L, H) *constitutes a SPE if and only if* $c^H < \theta^L q^H$*. Equilibrium prices are strictly above marginal costs.*

Proof. See the discussion above. A formal derivation can be found as a particular case of the proof to Proposition 7. ■

Propositions 1 and 2 form a remarkable result: arbitrarily small search frictions induce firms to switch from the "specialization" to the "overlapping" equilibrium, thus resulting in (weakly) lower prices.[27] This conclusion is particularly compelling when the costs of providing high quality are high enough, as in this case the "overlapping" equilibrium is unique. It is worth pointing out that this result is not driven by the rents created

[27]Using the terminology of Armstrong (2015), non-shoppers create a positive search externality to the shoppers, who end up paying lower prices.

by search frictions. Indeed, as this section has demonstrated, when search frictions are arbitrarily small such rents are close to zero and yet the "overlapping" equilibrium exists.

In contrast, in previous papers analyzing quality choices followed by imperfect competition (Gal-Or, 1983; Gilbert and Matutes, 1993; Johnson and Myatt 2003 and 2015; Stole, 1995; Wernerfelt, 1986), the "overlapping" equilibrium exists *only if* the rents created by imperfect competition are high enough (e.g., few firms competing *à la* Cournot, or price competition with sufficiently differentiated products). In those papers, just as in CR, there is a tension between competition and price discrimination: competition reduces the rents on the overlapping products at the same time as it enlarges consumers' information rents, thus reducing the gains from price discrimination.

In this paper, under the "overlapping" equilibrium that arises with search frictions, such a tension is not present because firms only care about the profits made out of the non-shoppers, out of which they obtain monopoly profits (in expectation). Thus, firms' product choices are solely driven by their incentives to discriminate consumers, leading them to carry the full product range even when the rents created by search frictions are arbitrarily small. This shows that the impact of search frictions on product choices, and through these on prices, is fundamentally different as that of other forms of imperfect competition.

5 Equilibrium Product and Price Choices

In this section we characterize equilibrium product and price choices for all values of $\mu < 1$. We show that the "overlapping" equilibrium is robust to introducing search frictions, no matter how big or small. In contrast, the "specialization" equilibrium fails to exist when μ is sufficiently low or, for all μ, when c^H is sufficiently high. In general, the "overlapping" equilibrium is more likely to be unique the higher the level of search frictions and/or the higher the costs of providing high quality.

We again proceed by backwards induction by first analyzing equilibrium pricing behavior and then product choices. The pricing subgames will also serve to understand pricing decisions for non-overlapping product configurations, which may prove relevant to cases in which product choices are constrained by factors outside our model (e.g., fixed costs of carrying a product).

5.1 Pricing Behavior

We first provide an important property of pricing behavior by multi-product firms.

Lemma 1 *In equilibrium, multi-product firms choose incentive compatible prices for their products, i.e.,* $\Delta p \in \left[\theta^L \Delta q, \theta^H \Delta q\right]$.

Proof. See the Appendix. ■

Lemma above shows that it is always optimal for a multi-product firm to choose prices that satisfy incentive compatibility. The intuition is simple. If the price of the high quality product is too high so that all consumers buy the low quality product, it is profitable for the firm to reduce p^H, while leaving p^L unchanged, so as to attract the high types and obtain a larger profit margin. Similarly, if the price of the high quality product is too low so that all consumers buy it, it is profitable for the firm to increase p^H, while leaving p^L unchanged, so as to extract more surplus from the high types as these are willing to pay more for higher quality. This result constitutes an important departure from Varian (1980), as it implies that the price of one product cannot be picked independently from the price of another product *within* the same store.[28]

We are now ready to characterize equilibrium pricing at every possible subgame.

Full product overlap We start by considering subgames with full product overlap: (LH, LH), (L, L), and (H, H). The last two are similar to Varian's. Since single-product firms selling the same product are not constrained by incentive compatibility, they play a mixed strategy equilibrium with an upper bound equal to the (unconstrained) monopoly price. Under (L, L) all consumers are served, but under (H, H) the low types are left out of the market. We thus focus here on the remaining case (LH, LH) with full product overlap among multi-product firms.

Proposition 3 *Given product choices* (LH, LH)*, there does not exist a pure strategy equilibrium. The equilibrium must be in mixed strategies, and it must satisfy the following properties:*

(i) At the upper bound of the price support, firms choose the (constrained) monopoly prices, $\overline{p}^H = \theta^H q^H - \theta^L \Delta q$ *and* $\overline{p}^L = \theta^L q^L$*. Thus, at the upper bound, the high types' incentive compatibility constraint is binding,* $\Delta\overline{p} \equiv \overline{p}^H - \overline{p}^L = \theta^H \Delta q$.

(ii) At the lower bound of the price support, firms choose prices that are strictly above marginal costs, $\underline{p}^i > c^i$ *for* $i = L, H$*, and such that the high types' incentive compatibility constraint is not binding,* $\Delta\underline{p} \equiv \underline{p}^H - \underline{p}^L < \theta^H \Delta q$.

Proof. See the Appendix. ■

[28] This is in contrast to Johnson and Myatt (2015) prediction. In a model of quality choice followed by Cournot competition, they find conditions under which the equilibrium prices chosen my multi-product oligopolists are close to the single-product prices.

The non-existence of pure strategy equilibria is shared with most models of simultaneous search, starting with Varian (1980). It stems from firms' countervailing incentives, as on the one hand they want to reduce prices to attract the shoppers, but on the other, they want to extract all rents from the non-shoppers.

Despite this similarity, our analysis shows that equilibrium pricing by multi-product firms has a distinctive feature: it is constrained by incentive compatibility (Lemma 1). This comes up clearly when characterizing the upper bound of the price support: firms are not able to extract all the surplus from the non-shopper high types because firms have to give up information rents $\Delta\theta q^L$.

Since firms make strictly positive profits at the upper bound, prices at the lower bound must be strictly above marginal costs. The reduction in prices from the upper to the lower bound is more pronounced for the high quality product than for the low quality one. Competition for the high types is fiercer because selling the high quality product is more profitable. In turn, this implies that at the lower bound, the incentive compatibility constraint for the high types is not binding, so that the price wedge between the two products at the upper bound is wider than at the upper bound. We can conclude that high quality products are relatively cheaper during periods of "sales" *à la* Varian, i.e., when both goods are priced at the lower bounds of the price supports. Even when firms do not price the two goods simultaneously at the lower bound, the relative price difference never exceeds the one under monopoly, $\theta^H \Delta q$, as otherwise incentive compatibility would not be satisfied (Lemma 1). Thus, competition among multi-product firms reduces the relative prices of the two goods.

Since firms have to be indifferent between charging any price in the support, including the upper bounds, expected equilibrium profits are unambiguously given by

$$\Pi\left(LH, LH\right) = \frac{1-\mu}{2}\left[\lambda\pi^L + (1-\lambda)(\pi^H - \Delta\theta q^L)\right]. \tag{3}$$

Just as we noted in the previous section, these profits are a fraction $(1-\mu)/2$ of the (constrained) monopoly profits.

At the lower bound, each firm attracts all the shoppers plus its share of the non-shoppers of each type. Hence, expected profits can also be expressed as a function of the lower bounds,

$$\Pi\left(LH, LH\right) = \frac{1+\mu}{2}\left[\lambda(\underline{p}^L - c^L) + (1-\lambda)(\underline{p}^H - c^H)\right]. \tag{4}$$

Since there are two goods, and only one profit level, as defined in equations (3) and (4), the problem has an extra degree of freedom: there are potentially many price pairs $\underline{p}^L > c^L$ and $\underline{p}^H > c^H$ satisfying $\Delta\underline{p} < \theta^H \Delta q$ that yield the same equilibrium profits. This implies that, even though equilibrium profits are unique and well defined, there might be multiplicity of mixed strategy equilibria.

Because the incentive compatibility constraint of the high types is binding at the monopoly solution, a natural equilibrium to consider is one in which firms keep on pricing the low quality product as if they were just selling that product, but adjust their pricing for the high quality one. The following Lemma characterizes such an equilibrium:

Lemma 2 *Given product choices* (LH, LH)*, there exists a mixed-strategy equilibrium in which firms choose* p^L *in* $\left[\underline{p}^L, \overline{p}^L\right]$ *according to*

$$F^L(p^L) = \frac{1+\mu}{2\mu} - \frac{1-\mu}{2\mu}\frac{(\overline{p}^L - c^L)}{(p^L - c^L)}$$

and such that, for given p^L*, the price* p^H *is chosen in* $\left[\underline{p}^H, \overline{p}^H\right]$ *to satisfy*

$$\frac{p^H - c^H}{p^L - c^L} = \frac{\overline{p}^H - c^H}{\overline{p}^L - c^L} \tag{5}$$

where

$$\underline{p}^i = c^i + \frac{1-\mu}{1+\mu}(\overline{p}^i - c^i) > c^i,$$

and $\overline{p}^i$ *are the (constrained) monopoly prices, for* $i = L, H$*.*

Proof. See the Appendix. ■

The proposed equilibrium has several appealing features. While firms price the low quality product as if they were just selling that product (as in Varian's model), on average they choose lower prices for the high quality product than when they only sell that product. This is a direct implication of the fact that the firm cannot extract all the surplus of the non-shopper high types. Indeed, the resulting distribution for p^H,

$$F^H(p^H) = \frac{1+\mu}{2\mu} - \frac{1-\mu}{2\mu}\frac{(\overline{p}^H - c^L)}{(p^H - c^L)}$$

has the same functional form as in Varian. However, since the upper bound $\overline{p}^H$ is the constrained monopoly price, the whole distribution puts higher weight on lower prices all along the support than in the independent products case.

Under this equilibrium, the choice of p^L results in a unique choice of p^H such that the relative profit margin of the two products remains constant along the whole support; see equation (5).[29] In particular, the relative markups of the two products are the same

[29]Clearly, there exists another equilibrium with the same price supports and the same price distribution for good L but in which the firm randomizes the price of good H, given the choice of p^L, such that the two prices remain incentive compatible. Again, this multiplicity is inconsequential for the purposes of this analysis as all equilibria yield equal expected profits.

as under monopoly. That is, under this equilibrium, competition affects the *price levels* but not the *price structure* within the firm.[30]

The price difference that is embodied in this price structure can be expressed as

$$\Delta p = \kappa \theta^H \Delta q + (1 - \kappa) \Delta c.$$

Consistently with Lemma 1, the price difference is a weighted average between $\theta^H \Delta q$ (i.e., the price difference at the monopoly solution) and Δc (i.e., the price difference at the competitive solution), where the weight $\kappa = \left(p^L - c^L\right) / \left(\overline{p}^L - c^L\right)$ represents the distance to the upper bound. At the upper bound, when the incentive compatibility constraint of the high types is binding, the price difference is maximal, $\Delta \overline{p} = \theta^H \Delta q$. As we move down the support, the incentive compatibility constraint is satisfied with slack and the price difference narrows down. The difference is minimal at the lower bound, when $\kappa = (1 - \mu) / (1 + \mu)$. Importantly, as μ approaches one, the prices at the lower bound converge to marginal costs, and the price gap approaches Δc. The equilibrium would thus collapse to the competitive solution. On the other extreme, as μ approaches zero, the prices at the lower bound converge to monopoly prices so that the price gap approaches $\theta^H \Delta q$. The equilibrium would thus collapse to the monopoly solution.

Partial product overlap Let us now characterize equilibrium pricing in the subgames with partial overlap: (L, LH), (H, LH). Interestingly, even though the single-product firm does not face an incentive compatibility constraint *within* its store, its pricing is nevertheless affected by incentive compatibility considerations through the effect of competition *across* stores.

The following Proposition characterizes the price equilibrium at the (L, LH) subgame.

Proposition 4 *Given product choices* (L, LH)*:*

(i) A pure strategy equilibrium does not exist.

(ii) At the unique mixed-strategy equilibrium, firm LH *charges* $p^H = p^L + \theta^H \Delta q$*, and both firms choose* p^L *in* $\left[\underline{p}^L, \theta^L q^L\right]$*, with firm* L *putting a probability mass at the upper bound.*

Proof. See the Appendix. ■

[30] Note that in this equilibrium, the prices of the two products within a firm are positively correlated. This is in contrast to what the literature on multi-product loss-leading concludes. However, as discussed in the introduction, that literature applies to setups in which goods are complements and consumers buy more than one- in contrast to the assumptions made in this paper.

In equilibrium, the two firms choose random prices for the low quality product over a common support. In turn, given its price choice for the low quality good, the multi-product firm prices the high quality product to just comply with incentive compatibility for the high types. Hence, unlike the previous case, the price difference between the two products remains constant at $\theta^H \Delta q$ over the whole support, and the density of prices for the high quality product is the same as that for the low quality product (just shifted out to the right by $\theta^H \Delta q$). It follows that, whenever the multi-product firm has the low price for the low quality product, all the shoppers (both the low or the high types) buy from it. Otherwise, the single-product firm serves all the shoppers, including the low and the high types.

It is worth pointing out that the multi-product firm charges lower prices on average as compared to the single-product firm. The reason is that, when it has the low price, its ability to discriminate between the low and the high types allows the multi-product firm to make extra profits $\mu(1-\lambda)\varphi^H$ out of the shopper high types. Since the multi-product firm has stronger incentives to undercut its rival's price, the single product firm has to put a probability mass at the upper-bound. In turn, since the two firms cannot put a mass at the same price, it follows that when the single-product firm is pricing at the upper bound it is only selling to the non-shoppers with probability one. Importantly, this implies that the single-product firm's profits are a fraction $(1-\mu)/2$ of the monopoly profits

$$\Pi(L, LH) = \frac{1-\mu}{2}\pi^L.$$

Now we turn to characterizing the price equilibrium at the (H, LH) subgame.

Proposition 5 *Given product choices (H, LH), there exists $\widehat{\mu} \in (0,1)$ such that:*

(i) For $\mu \leq \widehat{\mu}$, there exists a unique pure strategy equilibrium: firm H chooses the (unconstrained) monopoly price $p^H = \theta^H q^H$, and firm LH chooses the (constrained) monopoly prices, $p^H = \theta^H q^H - \Delta\theta q^L$ and $p^L = \theta^L q^L$.

(ii) For $\mu > \widehat{\mu}$, there does not exist a pure strategy equilibrium. In the mixed-strategy equilibrium, firm LH chooses prices p^H in $\left[\underline{p}^H, \theta^H q^H - \Delta\theta q^L\right]$ with a mass on on its upper bound, and $p^L = \min\left\{\theta^L q^L, p^H - \theta^L \Delta q\right\}$. Firm H chooses prices p^H in $\left\{\left[\underline{p}^H, \theta^H q^H - \Delta\theta q^L\right], \theta^H q^H\right\}$ with a (strictly) positive mass on its upper bound.

Proof. See the Appendix. ■

There now exists a pure strategy equilibrium as long as the fraction of shoppers μ is small enough. At this equilibrium, the multi-product firm charges the (constrained) monopoly prices, while the single-product firm charges the (unconstrained) monopoly price for the high quality product. The single-product firm does not want to fight for the shoppers as it prefers to just serve the non-shoppers at the unconstrained monopoly price than fighting for the shoppers at the cost of leaving informational rents to the non-shoppers.

When the fraction of shoppers is higher, the above is no longer an equilibrium as it now pays the single-product firm to fight for the shoppers. In this case, the equilibrium must be in mixed strategies.[31,32] The precise shape of the mixed strategy equilibrium depends on whether it pays firm H to serve the low types or not.

If $c^H \geq \theta^L q^H$, it never pays firm H to serve the low types because the costs of high quality exceed their willingness to pay for it. Thus, the two firms compete for the shopper high types only, while the low quality product is still priced at the monopoly level, $\theta^L q^L$. Since the incentive compatibility constraint of the multi-product firm is not binding, its profits are the same as if the two products were sold independently. In contrast, when $c^H < \theta^L q^H$, the low types might be tempted to buy the high quality good when its price is sufficiently low. In this case, the price of the low quality good has to be reduced below the monopoly price to achieve separation. In particular, this implies that at the lower bound of the price support of firm LH, the price difference between the two goods falls down to $\Delta \underline{p} = \theta^L \Delta q < \Delta c$. In sum, even though the multi-product firm is a monopolist over the low quality good, competition with the rival's high quality good forces the firm to reduce the prices of both goods.

Regarding the single-product firm, since $\theta^H q^H - \Delta\theta q^L$ is the highest price that the multi-product firm would ever charge for the high quality good, the firm will play either the (unconstrained) monopoly price, $\theta^H q^H$, or something less than the (constrained) monopoly price, $\theta^H q^H - \Delta\theta q^L$. Any price in between is unprofitable, either because it doesn't extract enough from the non-shopper high types or because it doesn't attract the shoppers when the multi-product firm happens to price the good at or below $\theta^H q^H - \Delta\theta q^L$. In either case, profits remain as in the pure strategy equilibrium because $\theta^H q^H$ always belongs to the price support. Therefore, for all μ,

$$\Pi\left(H, LH\right) = \frac{1-\mu}{2}\left(1-\lambda\right)\pi^H,$$

again are a fraction $\left(1-\mu\right)/2$ of the monopoly profits.

Last, just as in the previous subgame, the equilibrium price distribution used by the multi-product firm for the high quality good (weakly) first-order stochastically dominates

[31]Interestingly, there is continuity between the pure and the mixed-strategy equilibrium. The two firms charge the upper bounds of their price supports, $\theta^H q^H - \Delta\theta q^L$ and $\theta^H q^H$, with positive and identical mass. This mass fades away as μ grows larger—from one, when $\mu \to \widehat{\mu}$ towards zero, when $\mu \to 1$.

[32]Unlike in subgame $\left(LH, LH\right)$, the equilibrium is now unique: since one firm only has one product, there are no longer two degrees of freedom as in the symmetric two product case.

that of the single-product firm. It follows that, on average, the price charged by the single-product firm for the high quality product exceeds the one charged by the multi-product firm.

Non-overlap Let us now move to characterizing equilibrium pricing in the subgames with no product overlap: $(\varnothing, L)$, $(\varnothing, H)$, $(\varnothing, LH)$ and (L, H). The first three correspond to the monopoly solution already characterized in Section 2.2 above. Hence, here we turn our attention to the more interesting subgame with specialized firms, (L, H).

Proposition 6 *Given product choices (L, H), there exists $\widetilde{\mu} \in (\widehat{\mu}, 1)$ such that:*

(i) For $\mu \leq \widehat{\mu}$, there exists a unique pure strategy price equilibrium: firms charge the (unconstrained) monopoly prices $p^H = \theta^H q^H$ and $p^L = \theta^L q^L$.

(ii) For $\mu > \widehat{\mu}$ there does not exist a pure strategy equilibrium. At the unique mixed-strategy equilibrium, firm L chooses prices p^L in $\left[\underline{p}^L, \theta^L q^L\right]$ with a mass on the upper bound. If $\mu \in (\widehat{\mu}, \widetilde{\mu})$ firm H chooses prices p^H in $\left\{\left[\underline{p}^H, \theta^H q^H - \Delta\theta q^L\right], \theta^H q^H\right\}$ with a mass on the upper bound that falls to zero as $\mu \to \widetilde{\mu}$; if $\mu \geq \widetilde{\mu}$, $\theta^H q^H$ is not part of firm H's support.

Proof. See the Appendix. ■

Equilibrium pricing at subgames (L, H) and (H, LH) share some similarities. In particular, just as in Proposition 5, if the mass of shoppers μ is small enough, there exists a pure strategy equilibrium as the firm selling the high quality product is better off serving the non-shopper high types at the (unconstrained) monopoly price than competing for the shopper high types.[33] Furthermore, there is continuity between the pure and the mixed strategy equilibrium in that the probability mass that the high quality firm puts on the (unconstrained) monopoly price fades away as μ grows larger.

The main difference between the two subgames is that, under (L, H), the high quality firm chooses not to include the unconstrained monopoly price in the support when μ is very large. The reason is that the profits from serving a small fraction of non-shoppers become lower than the profits from fighting for the shoppers.[34]

To illustrate this, note that the former converge to zero as $\mu \to 1$. However, the high quality firm's profits cannot be lower than its minmax, which is strictly positive as the firm can always make a profit margin of at least $(1 - \lambda)\,\varphi^H$ when selling the high quality

[33] Note that the threshold for the existence of a pure-strategy equilibrium is the same under both subgames.

[34] At subgame (H, LH), competition for good H is more intense given that both firms carry it. This explains why in that case firm H always puts mass at the unconstrained monopoly price, while at subgame (H, L) firm H eventually decides not to include it in its price support.

product to the shopper high types.[35] Similarly, the profits of firm L are (weakly) above its minmax as firm H is pricing above its costs with probability one; the comparison is weak only when $c^H \geq \theta^L q^H$: in this case, since firms are not competing for the low types, firm L makes monopoly profits in equilibrium, just as under its minmax.

5.2 Quality Choices

We are now ready to analyze product line decisions given the continuation equilibria characterized above.

Proposition 7 *Assume $\mu < 1$.*

(i) The "overlapping" equilibrium (LH, LH) constitutes a SPE for all $\mu < 1$.

(ii) There exists μ^* implicitly defined by

$$(1-\mu^*)(1-\lambda)\left(\pi^H - \pi^L\right) = (1+\mu^*)\lambda\left(\pi^L - \varphi^L\right)$$

such that the "specialization" equilibrium (L, H) constitutes a SPE if and only if $\mu \geq \mu^*$. The critical threshold μ^* is increasing in c^H.

(iii) The "overlapping" equilibrium is unique if and only if $\mu < \mu^*$.[36] *A sufficient condition for uniqueness is $c^H \geq \theta^L q^H$.*

Proof. See the Appendix. ■

In Proposition 2 we showed that a SPE with overlapping product lines arises as soon as we add an arbitrarily small amount of non-shoppers. Proposition 7 now shows that this prediction remains valid for all values of $\mu < 1$. The underlying logic remains the same: the existence of the "overlapping" equilibrium hinges upon the incentives of firms to mimic those of a monopolist, and this holds true as long as the mass of non-shoppers $(1-\mu)$ is positive, regardless of whether it is large or small.

Regarding the existence of the "specialization" equilibrium, Proposition 2 showed that it exists for $\mu \to 1$ as long as $c^H < \theta^L q^H$. Proposition 7 now shows that this prediction does not extend to all $\mu < 1$. In particular, whereas $c^H < \theta^L q^H$ is still needed, it is not sufficient: additionally, search frictions have to be sufficiently low for the gains

[35] This was not the case under product choices (H, LH): in that case, the single-product firm's minmax profits go down to zero as $\mu \to 1$, as all shoppers would buy product H from the multi-product firm if the single-product firm tried to raise its price above marginal cost.

[36] In the proof of the Proposition we also show that there exists $\mu^{**} \geq \mu^*$ such that if $\mu \geq \mu^{**}$, there also exists a MSE such that firms randomize between L and H. Since the condition for the existence of this equilibrium is more demanding than it is for the pure-strategy equilibrium (L, H), it still holds true that the overlapping equilibrium is unique for $\mu < \mu^*$.

from softening competition to exceed the costs of giving up profitable opportunities to discriminate. To see this in more detail, consider the incentives to deviate from the "specialization" equilibrium by the firm carrying product L. Adding product H would allow the firm to better discriminate the high types, thus making extra profits $\left(\pi^H - \pi^L\right)$ from selling product H to the non-shopper high types with probability $(1-\mu)(1-\lambda)/2$. In contrast, adding product H would also intensify competition for product L, forcing the firm to give up rents $\left(\pi^L - \varphi^L\right)$ on all the low types (excluding the non-shopper low types that visit the rival's store) with probability $(1+\mu)\lambda/2$. The magnitude of the two effects coincides at $\mu = \mu^*$, as implicitly defined in the statement of the Proposition. In turn, since in expectation firms only benefit from discriminating the non-shoppers, the softening of competition effect dominates the incentives to discriminate only when the mass of non-shoppers $(1-\mu)$ is sufficiently small, i.e., when $\mu \geq \mu^*$. Therefore, for $\mu < \mu^*$ the "specialization" equilibrium breaks down, making the "overlapping" equilibrium the unique SPE of the game.

The fact that μ^* is increasing in c^H means that, as the cost of high quality provision increases up to $\theta^L q^H$, the set of μ values for which the "overlapping" equilibrium is unique is enlarged; beyond that level, the "overlapping" equilibrium is the unique SPE for all μ. In case $c^H \geq \theta^L q^H$, $\mu^* = 1$. In words, for high c^H, the "specialization" equilibrium never exists in the presence of non-shoppers because if firm L adds product H it does not give up any rents on product L.

We remain agnostic as to which equilibrium firms are more likely to play when there exist multiple equilibria (i.e., for parameter values $c^H < \theta^L q^H$ and $\mu \geq \mu^*$). Still, we want to stress that there are theoretical reasons, beyond its empirical relevance, to believe that the "overlapping" equilibrium is compelling. First, the Pareto criterion does not allow to rule out the "overlapping" equilibrium in general, despite the fact that it results in lower prices. In particular, the firm that carries product H at the "specialization" equilibrium is not necessarily better off than at the "overlapping" equilibrium as at the former it fails to capture the profits from serving the non-shopper low types. Furthermore, some authors have documented path dependency in equilibrium choices (Romero, 2015). In this setting, this suggests that the existence of the "overlapping" equilibrium for all $\mu < 1$ (in contrast to the "specialization" equilibrium, which only exists for $\mu \geq \mu^*$), together with low μ as an initial condition, might create inertia at (LH, LH) all the way down to $\mu \to 1$.[37]

[37] Consider for instance a simple repetition of our two-stage game and allow for search frictions to gradually fall. If, as initial condition, there are strong search frictions so that the "overlapping" equilibrium is unique, hysteresis would lead firms to keep on playing the same equilibrium even if the reduction in search frictions implies that the "specialization" equilibrium eventually arises. The same would apply

5.3 Comparative Statics with respect to Search Frictions

Last, combining the results in Propositions 1, 3, and 7, our last Lemma evaluates the impact of search frictions on expected market prices and expected consumer surplus at the SPE product choices. Results are illustrated in Figures 1 and 2.

Lemma 3 *(i) Expected prices decrease in $\mu \in (0, 1)$, with an upwards discontinuity at either $\mu = 1$ or at $\mu = \mu^*$. Similarly, expected consumer surplus increases in $\mu \in (0, 1)$, with an downwards discontinuity at either $\mu = 1$ or at $\mu = \mu^*$.*[38]

(ii) There exists $\mu' \in (0, 1)$ such that expected prices are higher and expected consumer surplus is lower at $\mu = 1$ (i.e., there are no search frictions) than at $\mu \in (\mu', 1)$ (i.e., search frictions are positive but not too high) whenever, in case of multiplicity, firms play the "overlapping" equilibrium.

Proof. See the Appendix. ■

The conventional wisdom that milder search frictions lead to lower prices applies in this model, but only when the reduction in search frictions does not change equilibrium product lines. Indeed, when lower search frictions induce firms to switch from the "overlapping" to the "specialization" equilibrium (either at $\mu = 1$ or at $\mu = \mu^*$, depending on equilibrium selection), expected prices jump up as firms manage to mitigate competition through product choice. Hence, as long as search frictions are not too high (i.e., for $\mu > \mu'$) and as long as firms play the "overlapping" equilibrium (i.e., which is unique for $\mu < \mu^*$), expected prices are lower in a market with search frictions than in a frictionless market ($\mu = 1$).

Similarly, as search frictions go down, consumer surplus goes up with a discontinuity when firms switch from the "overlapping" to the "specialization" equilibrium.[39] The discontinuity in consumer surplus is more pronounced than the discontinuity in expected prices because not only expected prices jump up, but also gross consumers' surplus jumps down because of incomplete price discrimination at the "specialization"

if the costs of quality are initially high and declining. In contrast, if there are initially no search costs and the costs of quality provision are low, the market could remain at the "specialization" equilibrium as either search costs or quality costs go up, eventually giving rise to the "overlapping" equilibrium when the "specialization" equilibrium ceases to exist. However, given the overall current trend towards lower search costs, this possibility does not seem empirically relevant.

[38] Whether the discontinuity occurs at $\mu = 1$ or at $\mu = \mu^*$ depends on whether, in case of multiplicity, firms play either the "overlapping" or the "specialization" equilibrium, respectively.

[39] Note that consumers never incur search costs since the shoppers have zero search costs and the non-shoppers do not search. Hence, search costs do not enter directly into the expression of consumer surplus; only indirectly to the extent that they affect prices and product choice.

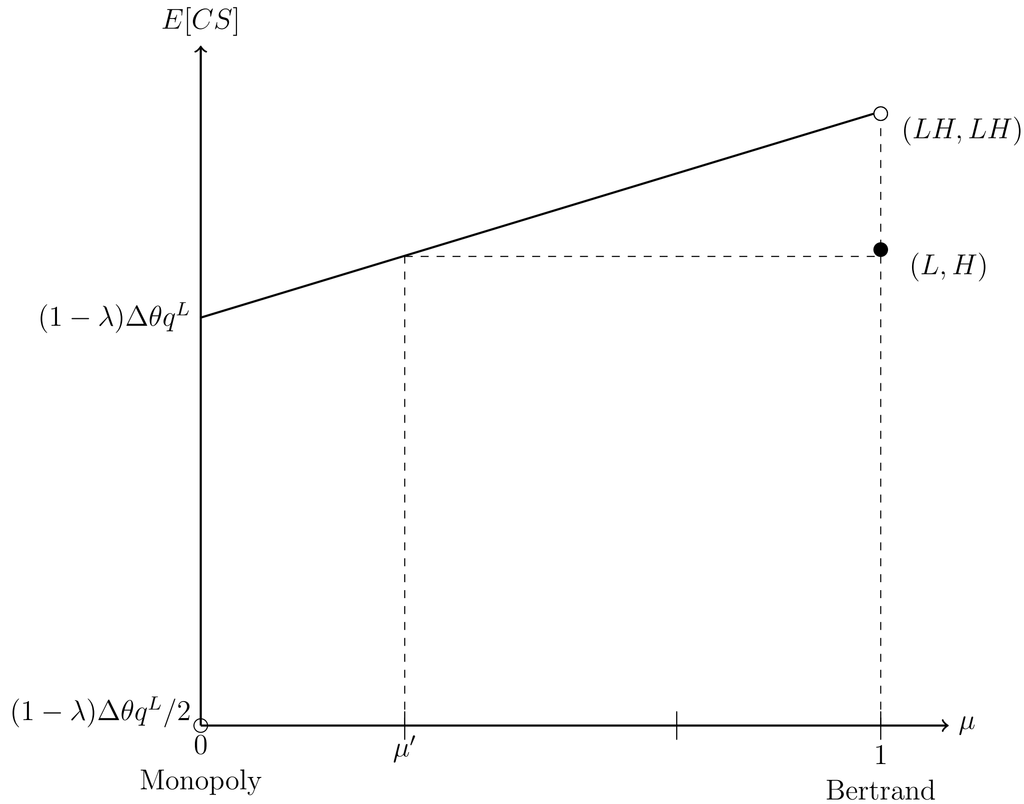

Figure 1: Expected consumer surplus as a function of μ at the SPE product choices for c^H above $\theta^L q^H$.

equilibrium (meaning that some high-types fail to buy their preferred good while some low-types fail to consume at all). Consumer surplus is the same at the "overlapping" equilibrium for μ' as compared to the "specialization" equilibrium for $\mu = 1$: in both cases, all consumers buy their preferred products and expected prices coincide. Hence, since consumer surplus at the "overlapping" equilibrium is increasing in μ, it also follows that consumers are better off when search frictions are positive but not too high (i.e., for $\mu > \mu'$) than when there are no search frictions at all ($\mu = 1$).

6 Extensions and Variations

In the preceding sections we characterized product and price choices in (*i*) a duopoly model, (*ii*) with two possible quality levels and two consumer types, in which (*iii*) search cannot be conditioned on product choices (as these were assumed non-observable prior to search), and in which (*iv*) consumers' search frictions and quality preferences are

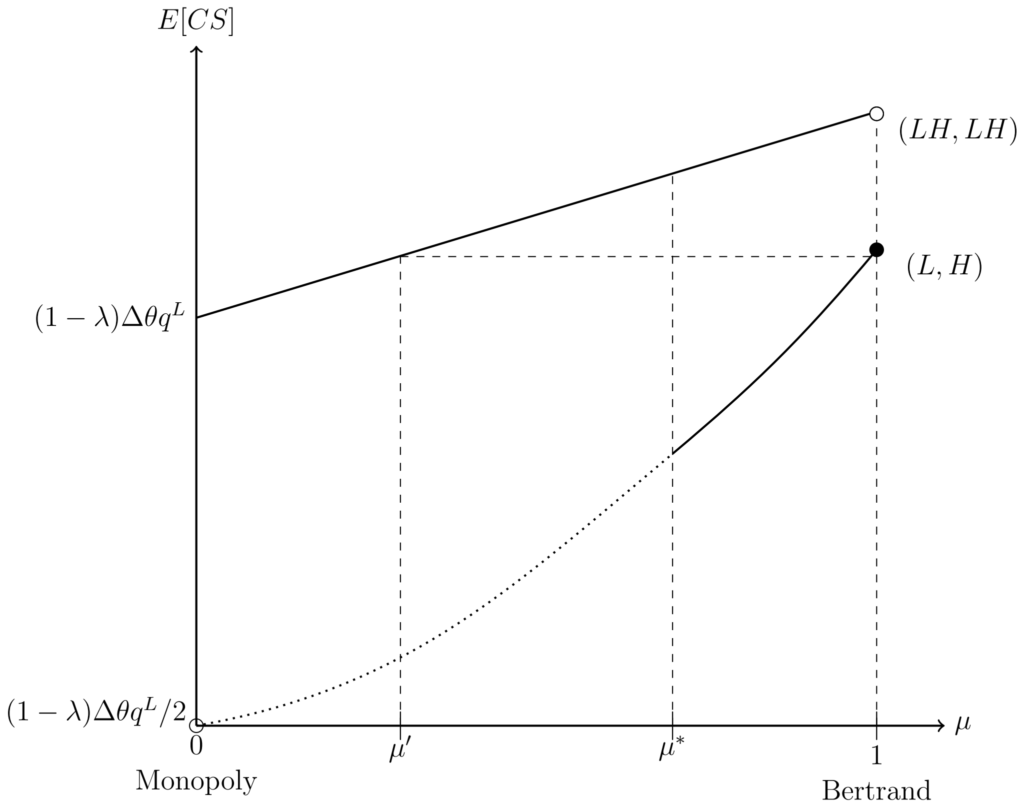

Figure 2: Expected consumer surplus as a function of μ at the SPE product choices for c^H below $\theta^L q^H$.

uncorrelated. In this section we discuss how one can relax each of these assumptions while preserving our main results. Our focus is on the existence of the "overlapping" equilibrium.

N **products,** N **consumer types** Consider the case in which there are N potential products of qualities $q^1 < ... < q^N$, and N consumer types with quality preferences $\theta^1 < ... < \theta^N$.

As we did above, let us start by characterizing the monopoly solution and by rephrasing our initial assumptions for the N products and N consumer types case. As it is well known, at the constrained monopoly solution, only the participation of the lowest quality product is binding, so that $p^1 = \theta^1 q^1$. For the remaining products, the constrained monopoly solution makes each consumer type indifferent between buying its preferred product or the one immediately below it in the quality ladder, i.e., $p^i = p^{i-1} + \theta^i \left(q^i - q^{i-1}\right)$. Thus, information rents accumulate across consumer types, and are therefore greater for higher types.

In the N-products case, assumption $(A1)$ can be restated as requiring the monopolist to find it optimal to carry all the products, even if this requires leaving information rents to some types. Assumption $(A2)$ can be restated by requiring that prices equal to marginal costs must induce full consumer separation. In other words, consumers of type i are willing to pay for the increase in quality from q^{i-1} to q^i, while consumers of type $i-1$ are not. Last, assumption $(A3)$ can be restated by making firms prefer to sell product i to the i-types at its constrained monopoly price, rather than reducing it to $\theta^{i-1}q^i$ so as to also serve lower type consumers.

Regarding the equilibrium for the case with no search frictions, it is straightforward to show that the same logic extends as before, making the equilibrium with non-overlapping product choices the unique SPE at which all firms make a positive profit (Proposition 1).

Regarding the case with positive search frictions, a similar logic as in the 2-products case also allows to conclude that both firms carrying all products constitutes a SPE for all $\mu < 1$ (Proposition 7). In the pricing stage, the upper bounds of the price supports in the mixed strategy equilibrium are the constrained monopoly prices given that at such prices the firm only sells to the non-shoppers (since firms are symmetric, there cannot be a mass at the upper bound, so when one firm prices at the upper bound, the other firm has strictly lower prices with probability one). Thus, equilibrium profits are a fraction $(1-\mu)/2$ of the constrained monopoly profits just as in the 2-product case (Proposition 3).

Alternatively, if one firm deviates by dropping one or more products, the deviant reduces its profits. To see this, first note that the upper bounds in the support of the firm with fewer products are equal to the constrained monopoly prices (conditional on the products it carries), which it must play with positive probability. Such a probability mass is in place to equate the profits of the rival firm at the upper and lower bounds; without the mass, the firm that carries a larger product line would have incentives to keep on reducing prices below the lower bound, as its increased ability to price discriminate allows it to make higher profits from attracting the shoppers. Since it is not possible for the two firms to put a mass at the upper bound, it follows that when the firm with fewer products prices at the upper bound, the rival firm must be charging strictly lower prices with probability one. Again, this implies that in equilibrium, the firm carrying fewer product makes a fraction $(1-\mu)/2$ of the constrained monopoly profits conditional on the products it carries. The proof concludes by noting that $(A1)$ guarantees that such profits are lower than the profits from carrying the complete product line. Therefore, as we have argued, the exact same results as in the 2-product extend to the N-product case.

N **products,** N **consumer types,** N **firm oligopoly** We can extend the previous discussion by allowing for N firms. As already argued, (i) Bertrand competition for overlapping products implies that the equilibrium when all consumers are shoppers has non-overlapping product lines; and (ii) the presence of shoppers restores the monopolist's incentives to carry the complete product line as each firm's profits are proportional to monopoly profits (the factor being $(1-\mu)/2$ under duopoly or, more generally, $(1-\mu)/N$ for the N-firms case). Since both forces are at play regardless of whether there are two or any arbitrary (finite) number of firms, the same result holds true.

Observable product choices and directed search by the non-shoppers In the main model we assumed that consumers do not observe product lines prior to visiting the stores. In particular, we assumed that the non-shoppers visit one of the two stores with equal probably, regardless of their product choices. Instead, suppose now that the non-shoppers visit the store that gives them higher expected utility, given firms' (observable) product choices and expected prices (in case of indifference, non-shoppers visit the store that carries their preferred product).[40] Allowing search to be conditioned on product choices would strengthen our main result: when directed search is allowed, carrying multiple products would allow firms to not only better discriminate, but also to attract more non-shoppers.

Directed search by the non-shoppers only affects pricing when firms have chosen

[40]This interpretation of non-shoppers as sophisticated buyers is closer to that in the clearing-house model *à la* Baye and Morgan (2001).

asymmetric product lines (with symmetric product lines, expected prices are also symmetric so it is irrelevant whether search is directed or random). Let us consider subgame (L, LH). Now, all the non-shopper high types visit the multi-product firm given that (i) the expected utility of buying product L is the same across the two stores, and (ii) at store LH they are indifferent between buying L or H. In turn, the prices for product L have to be such that the non-shopper low types are indifferent between visiting one store or the other (otherwise, they would all visit the one charging lower prices, but this cannot constitute an equilibrium as the high-priced firm would make no sales). From our previous analysis, we know that with an even split of non-shoppers between the two stores, the multi-product firm charges lower prices. Hence, to rebalance firms' pricing incentives, more than one half of the non-shopper low-types must visit store LH until their expected prices converge. Thus, since the market share of the single-product firm is lower, it makes lower profits than when product lines are non-observable, as we had assumed in the main model. In turn, this implies that firms have no incentives to deviate from (LH, LH) to (L, LH) - their incentives to deviate are weaker than in the main model, under which (LH, LH) already constituted an equilibrium for all $\mu < 1$ (Proposition 7). A similar reasoning applies to subgame (H, LH).

In sum, our main conclusion - namely, that the "overlapping" equilibrium is robust for all $\mu < 1$ - remains valid regardless of whether product lines are observable (and there is directed search by the non-shoppers) or not. The conclusion that multi-product firms tend to charge lower expected prices would have to be qualified, as with directed search firms charge the same expected prices even though multi-product firms make higher profits as they attract more customers.

Correlation between search frictions and quality preferences Last, we have so far assumed that shoppers and non-shoppers are equally likely to be either high or low types. However, this may not hold in practice. For instance, if low types are lower income consumers with more time to search, then non-shoppers are more likely to be high types. Alternatively, if high types enjoy shopping for their preferred (high quality) product, then non-shoppers are more likely to be low types. Ultimately, this is an empirical question whose answer may vary depending on the type of product or context considered. However, as far as the predictions of the model are concerned, it is inconsequential whether the correlation between search frictions and quality preferences is positive, negative or non-existent.

To formalize this, one can introduce the parameters λ^S and λ^{NS}, representing the fraction of low types among the shoppers and non-shoppers, respectively (if $\lambda^S > \lambda^{NS}$ there is positive correlation between search frictions and quality types as the fraction of low types is higher among the shoppers than among the non-shoppers, or negative otherwise). The analysis of product and price choices without search frictions remains intact since all consumers are shoppers by definition. As for the analysis with search frictions, profits on good H are proportional to $\left(1-\lambda^{NS}\right)$ and those on good L are proportional to λ^{NS}, thus implying that the incentive structure remains unchanged. As such, the "overlapping" equilibrium always exists just as in the case with no correlation between search and quality preferences.

7 Conclusions

In this paper we have analyzed a simple model of quality choice followed by price competition in markets with search frictions. We have found that an arbitrarily small amount of search frictions is enough to overturn the prediction that firms are always able to soften competition by carrying non-overlapping product lines, as in the seminal paper of Champsaur and Rochet (1989). Our results extend to more general settings, including the case with more than two goods and consumer types, more than two firms, directed search by the non-shoppers and the possibility that search frictions and quality tastes are positively or negatively correlated.

We have shown that, through product choice, search frictions have important implications for market outcomes beyond their well studied price effects. Furthermore, we have shown that analyzing the price effects of search frictions without endogenizing product choices can sometimes lead to overestimating the anticompetitive effects of search frictions. In particular, a small amount of search frictions can create head-to-head competition by inducing firms to carry overlapping products.

The multi-product nature of firms also adds important twists to the analysis of competition in the presence of search frictions. An important departure from Varian (1980) is that goods within a store cannot be priced independently from each other. In particular, the incentives to separate both consumer types impose an upper (lower) bound on the highest price that can be charged for a high (low) quality good, given the price of the low (high) quality one. This holds true even for a single-product firm competing with a multi-product one, as price discrimination within the latter spreads to the former through the effect of competition.

In line with Varian (1980),we have also shown that search frictions give rise to price dispersion when the two competing firms carry multiple products- a possibility not considered by Varian (1980). However, if one of the firms specializes in selling the high quality product, we have shown that a pure strategy equilibrium arises when search frictions are high enough, i.e., the dispersion prediction might no longer hold. In particular, the market might be segmented between the non-shoppers who visit the single-product high quality store, and the remaining consumers who pay lower prices at the multi-product store.

Admittedly, there are several motives other than the ones studied in this paper that shape firms' product choices. In particular, throughout the analysis we have assumed that firms do not incur any fixed cost of carrying a product. This modelling choice was meant to highlight the strategic motives underlying product choice. However, fixed costs of carrying a product (which could arguably be higher for high quality products),[41] could induce firms to offer fewer and possibly non-overlapping products. Our prediction is not that competitors should always carry overlapping product lines. Rather, our analysis suggests that if their product lines do not overlap in markets in which search frictions matter, it must be for reasons other than firms' attempts to soften competition through product choice- for instance, due to the presence of fixed costs.

To the extent that firms could collude to coordinate their product choices (as reported by Sullivan (2016) in the context of the super-premium ice cream market),[42] competition authorities should remain vigilant if competitors' product lines do not overlap - particularly so in markets in which fixed costs (at the product level) are not relevant but consumers find it costly to search.

Appendix: Proofs

Proof of Proposition 1 [SPE under $\mu = 1$] We show that the specialization equilibrium (L, H) constitutes a SPE. First, at subgames (LH, LH), (L, L) and (H, H) , both firms make zero profits. Second, at subgame (L, LH) the low quality product is priced at marginal cost c^L while the high quality product is sold at the highest price that satisfies the high types' incentive compatibility constraint, i.e., $c^L + \theta^H \Delta q$. Firm L makes zero

[41] In some cases, such costs can be substantial, e.g. firms have to advertise that they are carrying an additional product, or the transaction costs of dealing with an additional provider can sometimes be high. The marketing literature has analyzed several factors explaining the limited number of products sold per firm. For instance, Villas-Boas (2004) analyzes product line decisions when firms face costs of communicating about the different products they carry to their customers. They show that costly advertising can induce firms to carry fewer products as well as to charge lower prices for their high-quality goods.

[42] See also the *NY Times* note quoted in the paper. Although it is difficult to divide "smooth" and "chunky" flavors in low and high-quality options, the logic of our result may apply as well. The ice-cream company that focuses on chunky flavors may need to reprice its existing offer downwards if it decides to also carry smooth flavors and compete head-to-head on these flavors with the rival company just carrying them. But this is profitable as long as exist some fraction of smooth non-shoppers.

profits while firm LH gets a payoff of $(1-\lambda)\varphi^H$, which equals its minimax. Third, at subgame (H, LH), the high quality product is priced at marginal cost c^H while the low quality product is sold at the highest price that satisfies the low types' incentive compatibility constraint and participation constraints, i.e., $\min\left\{c^H - \theta^L \Delta q, \theta^L q^L\right\}$. Firm H makes zero profits while firm LH makes profits $\lambda\pi^L$ if $c^H > \theta^L q^H$ or $\lambda\varphi^L$ otherwise, i.e., its minmax. Finally, at subgame (L, H) the equilibrium is in mixed strategies. For the purposes of this proof, it suffices to put bounds on equilibrium profits. Minmax profits for each firm are computed by characterizing the firm's best response to the rival pricing its good at marginal cost. Following our previous analysis, the minmax profits for the H firm are $(1-\lambda)\varphi^H > 0$, while the minmax profits for the L firm are $\lambda\pi^L > 0$ if $c^H > \theta^L q^H$ or $\lambda\varphi^L > 0$ otherwise. Since at the mixed strategy equilibrium firms always price above marginal costs (otherwise they would have zero profits, but this cannot be since their minmax profits are positive), equilibrium profits are *strictly above* the minimax whenever the participation constraint is not binding. The only case where above marginal cost pricing does not necessarily imply that firm L's profits are strictly above its minmax is when $c^H > \theta^L q^H$, as in this case firm L's best response is the the same regardless of whether firm H prices at c^H or above.[43] Indeed, for the case $c^H > \theta^L q^H$, equilibrium profits are exactly equal to the minmax $\lambda\pi^L$. To see this, note that at the MSE the upper bounds of firms' price supports are the constrained monopoly prices. Furthermore, firm L has to play a probability mass at its upper bound. Otherwise, firm H would make zero profits at its upper bound (as all consumers would strictly prefer to buy from firm L), but this cannot be the case since its minmax is strictly positive. Last, the two firms cannot put positive mass at their upper bounds as firm L would be better off putting all its mass slightly below its upper bound (so as to attract all consumers whenever the rival plays the mass at the upper bound). It thus follows that when firm L plays its upper bound, the rival is pricing below its upper bound with probability one. Hence, at the upper bound firm L only serves the low types, thus making profits that exactly equal its minmax, $\lambda\pi^L$.

We are now ready to show that (L, H) is a SPE. Starting at (L, H), firm H does not want to carry good L as at (L, LH) its profits are equal to the minmax, while they are

[43] It is straightforward to see that in a mixed strategy equilibrium we must have $\underline{p}^H > c^H$ and $\underline{p}^L > c^L$; otherwise, each firm's profits would be zero, but this leads to a contradiction since profits cannot be below the minimax. Hence, firm H would never like to price lower than $\underline{p}^L + \theta^H \Delta q > c^L + \theta^H \Delta q > c^H$. Since at a price $\underline{p}^L + \theta^H \Delta q$ firm H would at least be serving the high types, its profits must be strictly greater than its minmax $(1-\lambda)\varphi^H$. Similarly, if $\underline{p}^H > \theta^L q^H$, firm L would be a monopolist over the low-types, so it could always secure profits of at least $\lambda\pi^L$. If $\underline{p}^H < \theta^L q^H$, firm L would never like to charge prices lower than $\underline{p}^H - \theta^L \Delta q > c^H - \theta^L \Delta q$. Since at a price $\underline{p}^H - \theta^L \Delta q$ firm L would at least be serving the low types, its profits must be strictly greater than its minmax $\lambda\varphi^L$.

strictly above that level at (L, H). Similarly, firm L does not want to carry good H as at (LH, H) its profits are equal to the minmax, while at (L, H) its profits are (weakly) greater than its minmax.

Last, we characterize the MSE. Suppose that the rival chooses L with probability α and H with probability $(1-\alpha)$. Equating the profits from choosing L and H,

$$\alpha\Pi(L, L) + (1-\alpha)\Pi(L, H) = \alpha\Pi(H, L) + (1-\alpha)\Pi(H, H)$$

Since $\Pi(H, H) = \Pi(L, L) = 0$, solving for α,

$$\alpha = \frac{\Pi(L, H)}{\Pi(L, H) + \Pi(H, L)}.$$

Thus implying that equilibrium profits at the MSE are

$$\frac{\Pi(L, H)\Pi(H, L)}{\Pi(L, H) + \Pi(H, L)}.$$

This equilibrium constitutes a SPE if and only if it is not dominated to choosing LH, i.e.,

$$\Pi(L, H)[\Pi(L, L) - \Pi(LH, L)] + \Pi(H, L)[\Pi(L, H) - \Pi(LH, H)] \geq 0$$

The first term is negative, while the sign of the second term depends on c^H : (i) if $c^H \geq \theta^L q^H$, it is negative, implying that LH dominates the candidate MSE, which therefore does not exist; on the contrary, (ii) if $c^H < \theta^L q^H$, the second term is positive, implying that a MSE cannot be ruled out, particularly so for low c^H, which is when the second term is higher (note that as $c^H \to \theta^L q^H$ the second term is close to zero, so the MSE is ruled out for some $c^H < \theta^L q^H$). Therefore, for those parameter values for which the above inequality holds, a MSE exists. *Q.E.D.*

Proof of Proposition 2 [SPE under $\mu \to 1$] The results on existence and uniqueness of the "overlapping" equilibrium (LH, LH) under the assumption $\mu \to 1$ are a particular case of the proof of Proposition 7. The proof of non-existence of the "specialization" equilibrium (L, H) for $\mu \to 1$, for the case $c^H \geq \theta^L q^H$ is also contained in the proof of Proposition 7. Hence, it only remains to prove that $c^H < \theta^L q^H$ implies the existence of the "specialization" equilibrium. The proof of Proposition 1 above shows that $\Pi(L, H)$ and $\Pi(H, L)$ are *strictly* above the minmax, while the proofs of Proposition 5 and 4 show that $\Pi(LH, H)$ and $\Pi(LH, L)$ are equal to their minmax as $\mu \to 1$. It follows that no firm wants to deviate from (L, H). *Q.E.D.*

Proof of Lemma 1 Argue by contradiction and suppose that the firm chooses $\Delta p > \theta^H \Delta q$. Hence, all buyers visiting the store buy product L, and the firm makes a profit margin equal to $\left(p^L - c^L\right)$. If the firm reduced p^H so that $\Delta p = \theta^H \Delta q$, it would still sell product L to the low types at the same price, but would now sell product H to the high types with a higher profit margin $p^H - c^H = p^L + \theta^H \Delta q - c^H > p^L - c^L$, where the inequality follows from $(A2)$. A similar reasoning applies to rule out $\Delta p < \theta^L \Delta q$. $Q.E.D.$

Proof of Proposition 3 [pricing at subgame (LH, LH)**]** The non-existence of a pure strategy equilibrium follows from standard arguments. Firms cannot tie in prices as a slight reduction in the price would allow a firm to attract all the shoppers. Firms cannot charge different prices either as the high-priced firm would only serve the non-shoppers and would thus be better off by either undercutting the rival's price or by charging the (constrained) monopoly prices to maximize profits out of the non-shoppers; in turn, if the high-priced firm acts as the (constrained) monopolist, the other firm would find it profitable to slightly price below that level, thus not making it profitable any more for the rival to charge the (constrained) monopoly prices. Thus, the equilibrium must be in mixed-strategies. Since firms are symmetric, we focus on characterizing the symmetric mixed strategy equilibria. Standard arguments imply that there are no holes in the support and that firms play no mass point at any price of the support, including the upper bound (see, for instance, Narasimhan, 1998). (i) At the upper bound, firms serve the non-shoppers only. Since profits are increasing in prices subject to $\left(IC^H\right)$, the optimal prices at the upper bounds are $\overline{p}^H = \theta^H q^H - q^L \Delta\theta$ and $\overline{p}^L = \theta^L q^L$, so that $\Delta\overline{p} = \theta^H \Delta q$.

We now demonstrate (ii), i.e., that at the lower bound $\Delta \underline{p} < \theta^H \Delta q$. Suppose otherwise that the price gap $p^H - p^L$ is constant and equal to $\theta^H \Delta q$ at and in the neighborhood of the lower bound (or throughout the entire price support for that matter). When a firm plays $\underline{\mathbf{p}} = (\underline{p}^H, \underline{p}^L)$ it obtains

$$\Pi(LH, LH; \underline{\mathbf{p}}) = \left(\mu + \frac{1-\mu}{2}\right)(1-\lambda)(\underline{p}^H - c^H) + \left(\mu + \frac{1-\mu}{2}\right)\lambda(\underline{p}^L - c^L).$$

Using $\Pi(\cdot; \underline{\mathbf{p}}) = \bar{\pi} \equiv (1-\mu)\left[\pi^L + (1-\lambda)\left(\theta^H \Delta q - \Delta c\right)\right]/2$, the payoff at the upper bound, and the assumption that $\underline{p}^H - \underline{p}^L = \theta^H \Delta q$ we obtain

$$\underline{p}^H - c^H = \frac{1-\mu}{1+\mu}(\bar{p}^H - c^H) + \lambda \frac{2\mu}{1+\mu}\varphi^H \tag{6}$$

and

$$\underline{p}^L - c^L = \frac{1-\mu}{1+\mu}(\bar{p}^L - c^L) - (1-\lambda)\frac{2\mu}{1+\mu}\varphi^H. \tag{7}$$

We now compute the cdf $F(p^H)$ firms use in equilibrium to randomize prices. First, notice that if one firm plays something in the support, the other firm never wants to deviate and serve just the high type with a price $\theta^H q^H$, because according to $(A1)$ the payoff of doing so would be strictly lower. Thus, to obtain the cdf $F(p^H)$ around the lower bound, notice that playing any pair p^H and $p^L = p^H - \theta^H \Delta q$ around the lower bound yields an expected payoff of

$$\begin{aligned}\Pi(\cdot; p^H, p^L) &= (1-\lambda)(p^H - c^H)\left[\frac{1-\mu}{2} + \mu(1 - F(p^H))\right] + \\ &\quad \lambda(p^L - c^L)\left[\frac{1-\mu}{2} + \mu(1 - F(p^H))\right]\end{aligned}$$

where $1 - F(p^H)$ is the probability to attract both high and low type shoppers. Rearranging terms and using $\Pi(p^H, p^L = p^H - \theta^H \Delta q) = \bar{\pi}$ leads to

$$(\bar{p}^H - p^H)\frac{1-\mu}{2} = \left[1 - F(p^H)\right]\left[\mu(p^H - c^H) - \lambda\mu\varphi^H\right]. \tag{8}$$

From this expression we obtain

$$f(\bar{p}^H) = \frac{1-\mu}{2[\mu(\bar{p}^H - c^H) - \lambda\mu\varphi^H]} > 0$$

and

$$f(\underline{p}^H) = \frac{1+\mu}{2[\mu(\underline{p}^H - c^H) - \lambda\mu\varphi^H]} > 0. \tag{9}$$

Since $\underline{p}^H - c^H = \underline{p}^L - c^L + \varphi^H$, from (9) we also obtain that

$$f(\underline{p}^L) = \frac{1+\mu}{2[\mu(\underline{p}^L - c^L) + (1-\lambda)\mu\varphi^H]} > 0. \tag{10}$$

If the lower bound $\underline{\mathbf{p}}$ is indeed part of the equilibrium support, firms would not want to deviate from it. There are four possible (local) deviations to consider: (i) $\underline{p}^L$ and $p^H = \underline{p}^H - \varepsilon$, (ii) $\underline{p}^L$ and $p^H = \underline{p}^H + \varepsilon$, (iii) $\underline{p}^H$ and $p^L = \underline{p}^L - \varepsilon$, and (iv) $\underline{p}^H$ and $p^L = \underline{p}^L + \varepsilon$, where $\varepsilon \to 0$. The first deviation is clearly not profitable. If the firm plays $p^H = \underline{p}^H - \varepsilon$, it sells the same amount but at a lower price. Playing (ii) $p^H = \underline{p}^H + \varepsilon$ is also unprofitable. It violates the IC for high type consumers; so the firm would end up selling only low quality products to both, all shoppers and the non-shoppers coming to the store. Deviation (iii) is also unprofitable for the same reason (ii) is. We are left with deviation (iv). Notice first that playing $\underline{p}^H$ and $p^L = \underline{p}^L + \varepsilon$ only affects profits from low type consumers. The change in profit is

$$\Delta\Pi = \lambda\frac{1-\mu}{2}\varepsilon + \lambda\mu\left[(1 - F(\underline{p}^L + \varepsilon))(\underline{p}^L + \varepsilon - c^L) - (\underline{p}^L - c^L)\right].$$

The first term captures the gain from non-shoppers and the term in brackets captures the trade-off of losing the shoppers and charging them a bit more. We now take the derivative of $\Delta\Pi$ with respect to ε and evaluate it at $\varepsilon = 0$ to obtain

$$\left.\frac{\partial\Delta\Pi}{\partial\varepsilon}\right|_{\varepsilon=0} = \frac{\lambda(1-\mu)}{2} + \lambda\mu[1 - f(\underline{p}^L)(\underline{p}^L - c^L)]. \tag{11}$$

Replacing $f(\underline{p}^L)$ that follows from (10) into (11) we obtain $\partial\Delta\Pi/\partial\varepsilon|_{\varepsilon=0} > 0$, which contradicts that playing $\underline{\mathbf{p}}$ was an equilibrium. *Q.E.D.*

Proof of Lemma 2 [pricing at subgame (LH, LH)**]** We want to show that the equilibrium in the statement of the Proposition is indeed an equilibrium. First, firms could deviate by playing the price pairs in the support with different probabilities, while still choosing price pairs that satisfy incentive compatibility. However, this is unprofitable given that all price-pairs in the support give equal expected profits. Indeed, the equilibrium has been constructed so that

$$(p^L - c^L)\left[\frac{1-\mu}{2} + \mu(1 - F^L(p^L))\right] = \frac{1-\mu}{2}(\overline{p}^L - c^L) = \frac{1+\mu}{2}(\underline{p}^L - c^L)$$

and

$$(p^H - c^H)\left[\frac{1-\mu}{2} + \mu(1 - F^H(p^H))\right] = \frac{1-\mu}{2}(\overline{p}^H - c^H) = \frac{1+\mu}{2}(\underline{p}^H - c^H)$$

with the ratio (5) derived in order for the price pair $\left(p^H, p^L\right)$ to satisfy $F^H(p^H) = F^L(p^L)$, i.e., the choice of p^L results in a choice of p^H, so that the prices satisfying that ratio are played with equal probability. Therefore, expected profits at the proposed equilibrium are as in (3).

Second, firms could deviate by choosing p^L and p^H not satisfying equation (5) while still satisfying incentive compatibility. Again, these deviations are not profitable since all the prices in the support give equal profits. Deviating to prices that do not satisfy incentive compatibility is unprofitable because of Lemma 1.

Last, firms could deviate by playing price pairs outside the support. Choosing any prices above $\left(\overline{p}^L, \overline{p}^H\right)$ as defined above is unprofitable, as at these prices the firm is only selling to the non-shoppers and $\left(\overline{p}^L, \overline{p}^H\right)$ are the optimal monopoly prices. Choosing any prices below $\left(\underline{p}^L, \underline{p}^H\right)$ as defined above is unprofitable, as at these prices the firm is inelastically selling to all consumers with probability one and would thus gain by rasing the price up to $\left(\underline{p}^L, \underline{p}^H\right)$.

Proof of Proposition 4 [pricing at subgame (L, LH)**]** It is easy to see that the equilibrium must be in mixed strategies. Since both firms are competing to attract the shoppers, any PSE candidate can be ruled out by either firm's incentives to slightly undercut its rival. Note that by $(A1)$ it does not pay firm LH to only serve the high types. Both firms choose p^L in $\left[\underline{p}^L, \theta^L q^L\right]$ and incentive compatibility restricts firm LH to price the high quality product at $p^H = p^L + \theta^H \Delta q$.

We now show, by contradiction, that one of the two firms must be placing an atom at the upper bound. Suppose not, in which cases upper-bound payoffs are given by[44]

$$\Pi\left(L, LH; \bar{\mathbf{p}}\right) = \frac{1-\mu}{2}\pi^L \tag{12}$$

$$\Pi\left(LH, L; \bar{\mathbf{p}}\right) = \frac{1-\mu}{2}\left[\pi^L + (1-\lambda)\varphi^H\right]. \tag{13}$$

In the absence of atoms, when firm L prices at the upper bound, the shoppers low and all high types buy from the rival because with probability one firm LH would be pricing both goods at a lower price (after controlling for quality), whereas when LH prices at the upper bound, both shoppers' types prefer to buy the rival's low quality product. With these hypothetical "upper-bound" payoffs we can now find the lower bound of the price support $\underline{p}^L$, which must be atomless to rule out any deviation. If we equalize (12) to what firm L would get by pricing at the lower bound and attracting all shoppers and half of the non-shopper low and high types, i.e.,

$$\Pi\left(L, LH; \underline{p}^L\right) = \frac{1+\mu}{2}(\underline{p}^L - c^L).$$

We obtain

$$\underline{p}^L_l - c^L = \frac{1-\mu}{1+\mu}\pi^L \tag{14}$$

where the subindex l indicates that $\underline{p}^L_l$ is obtained using L's indifference condition. Similarly, if we equalize (13) to what firm LH would get by pricing at the lower bound and attracting all shoppers as well as non-shopper high types, i.e.,

$$\Pi\left(LH, L; \underline{\mathbf{p}}\right) = \frac{1+\mu}{2}\left(\underline{p}^L - c^L + (1-\lambda)\,\varphi^H\right).$$

We obtain

$$\underline{p}^L_{lh} - c^L = \frac{1-\mu}{1+\mu}\pi^L - \frac{2\mu}{1+\mu}(1-\lambda)\varphi^H \tag{15}$$

where the subindex lh indicates that $\underline{p}^L_{lh}$ is obtained using LH's indifference condition. Importantly, note that $\underline{p}^L_{lh} - c^L$ is decreasing in φ^H. Simple inspection of (14) and (15) shows that $\underline{p}^L_l \neq \underline{p}^L_{lh}$, except for $\mu = 0$, which contradicts the initial assumption that no firm places an atom at the upper bound for any $\mu > 0$. Furthermore, we have $\underline{p}^L_l > \underline{p}^L_{lh}$ for all $\mu > 0$. This implies that $\underline{p}^L_l$ is the actual lower bound and that firm L must be playing the upper bound with positive probability mass, while firm LH plays no mass at any price. The size of that mass ω^l, can be obtained by invoking LH's indifference

[44] Note that firm L serves the non-shoppers, both the low and the high types at $\theta^L q^L$.

condition[45]

$$\frac{1+\mu}{2}\left(\left(\underline{p}^{L}-c^{L}\right)+(1-\lambda)\,\varphi^{H}\right)=\left(\frac{1-\mu}{2}+\mu\omega^{l}\right)\left(\pi^{L}+(1-\lambda)\varphi^{H}\right)\right)$$

and after replacing $\underline{p}_{l}^{L}-c^{L}$ from (14),

$$\omega^{l}=\frac{1+\mu}{2\mu}\frac{\frac{1-\mu}{1+\mu}\pi^{L}+(1-\lambda)\,\varphi^{H}}{\pi^{L}+(1-\lambda)\varphi^{H}}-\frac{1-\mu}{2\mu}.$$

Equilibrium profits at this MSE are

$$\begin{aligned}\Pi\,(L,LH) &= \frac{1-\mu}{2}\pi^{L}\\ \Pi\,(LH,L) &= \frac{1+\mu}{2}\left(\frac{1-\mu}{1+\mu}\pi^{L}+(1-\lambda)\,\varphi^{H}\right).\end{aligned}$$

The characterization of the MSE ends with the probability distribution $F(p^L)$ used by the firms to set prices in the range $[\underline{p}^L, \theta^L q^L)$. Since both firms play according to the same distribution, one can derive such distribution from either firm's indifference condition. *Q.E.D.*

Proof of Proposition 5 [pricing at subgame (H, LH)] At the candidate PSE firm H charges $p^H = \theta^H q^H$ while firm LH charges $p^L = \theta^L q^L$ and $p^H = \theta^H q^H - \Delta\theta q^L$. At these prices firm H only sells to the non-shopper high types and firm LH to all the rest.[46] Thus, firms' profits at this PSE are

$$\Pi\,(H,LH) = \frac{1-\mu}{2}\,(1-\lambda)\,\pi^{H} \tag{16}$$

$$\Pi\,(LH,H) = \frac{1+\mu}{2}\left[\lambda\pi^{L}+(1-\lambda)(\pi^{H}-\Delta\theta q^{L})\right]. \tag{17}$$

For this to be an equilibrium, neither firm would want to deviate from it. While this is evident for firm LH, firm H could charge slightly less than $\theta^H q^H - \Delta\theta q^L$ to also attract the high type shoppers (note that at this price there are still unserved low types).[47] If it does, it would sell to all the high types except to the non-shoppers who buy from firm LH. It would thus make profits

$$\Pi'\,(H,LH)=\frac{1+\mu}{2}\,(1-\lambda)\,(\pi^{H}-\Delta\theta q^{L}). \tag{18}$$

[45] Notice that when firm L plays the atom $\theta^L q^L$, which happens with probability ω^l, firm LH's upper bound for p^L is not $\theta^L q^L$ but $\theta^L q^L - \varepsilon$ with $\varepsilon \to 0$. The reason is that in equilibrium there cannot be a tie on a price with positive probability.

[46] Recall from $(A2)$ that a firm carrying product H always prefers to serve just the high types rather than having to reduce prices to also serve the low types. Allowing for this possibility would duplicate the number of cases to consider without adding any qualitatively different results.

[47] Note also that this deviation implicitly uses $(A2)$, in particular, that serving the high types at price $\theta^H q^H - \Delta\theta q^L$ is more profitable for firm H than serving all consumers at price $\theta^L q^H$.

Comparing the profit expressions for firm H, this deviation is unprofitable if

$$\mu \leq \widehat{\mu} \equiv \frac{q^L \Delta\theta}{\pi^H + (\pi^H - \Delta\theta q^L)} < 1. \tag{19}$$

Hence, the PSE exists if and only if $\mu \leq \widehat{\mu}$. Otherwise the equilibrium must be in mixed strategies.

We now characterize the MSE. The upper bounds for firm LH must be $\overline{p}^L = \theta^L q^L$ and $\overline{p}^H = \theta^H q^H - \Delta\theta q^L$. For firm H, the upper bound could be either be the unconstrained monopoly price $\theta^H q^H$ or the constrained monopoly price $\theta^H q^H - q^L \Delta\theta$, but never something in between. We introduce ω^h and ω^{lh} to denote the masses that firms H and LH place at the upper bounds, respectively. As for the lower bounds, we also need to consider two possibilities, either $\underline{p}^H \geq \theta^L q^H$ or $\underline{p}^H < \theta^L q^H$. We go over these four possible cases next.

Cases 1 and 2: $\underline{p}^H \geq \theta^L q^H$ In these two cases, the low types never want to buy good H, so the $\left(IC^L\right)$ is never binding. Thus, firm H never serves the low types and firm LH can always price good L at the monopoly price $\theta^L q^L$.

Case 1: $\underline{p}^H \geq \theta^L q^H$ and the upper bound for firm H is $\theta^H q^H$ Firm H's profits at the lower and upper bounds are, respectively

$$\begin{aligned} \Pi\left(H, LH; \overline{\mathbf{p}}\right) &= \frac{1-\mu}{2}(1-\lambda)\pi^H \\ \Pi(H, LH; \underline{\mathbf{p}}) &= \frac{1+\mu}{2}(1-\lambda)\left(\underline{p}^H - c^H\right). \end{aligned}$$

Since these two have to be equal, it follows that

$$\underline{p}^H = c^H + \frac{1-\mu}{1+\mu}\pi^H. \tag{20}$$

Firm H must also be indifferent between playing $\theta^H q^H$ (and serving only the non-shopper high types) and $\theta^H q^H - \Delta\theta q^L - \epsilon$, with $\epsilon \to 0$ (and also serving shopper high types with some probability). If ω^{lh} is the probability mass that firm LH places at the upper bound, then

$$\frac{1-\mu}{2}(1-\lambda)\pi^H = \left(\frac{1-\mu}{2} + \mu\omega^{lh}\right)(1-\lambda)(\theta^H q^H - \Delta\theta q^L - c^H)$$

which yields

$$\omega^{lh} = \frac{1-\mu}{2\mu}\frac{\Delta\theta q^L}{\pi^H - \Delta\theta q^L}. \tag{21}$$

Note that the mass is decreasing in μ: as $\mu \to 1$, $\omega^{lh} \to 0$, and as $\mu \to \widehat{\mu}$, $\omega^{lh} \to 1$; hence, there is continuity between the PSE and the MSE.

On the other hand, whenever firm LH prices the two goods at the upper bounds, $\overline{p}^L = \theta^L q^L$ and $\overline{p}^H = \theta^H q^H - \Delta\theta q^L$, it obtains

$$\Pi\left(LH, H; \bar{\mathbf{p}}\right) = \frac{1+\mu}{2}\lambda\pi^L + (1-\lambda)\left(\frac{1-\mu}{2} + \mu\omega^h\right)\left(\pi^H - \Delta\theta q^L\right)$$

which must be equal to what it gets by playing at $\underline{p}^L = \theta^L q^L$ and $\underline{p}^H$, as defined in (20),

$$\Pi\left(LH, H; \underline{\mathbf{p}}\right) = \left(\frac{1-\mu}{2} + \mu\right)\lambda\pi^L + \left(\frac{1-\mu}{2} + \mu\right)(1-\lambda)(\underline{p}^H - c^H).$$

Equalizing these two latter expressions implies that $\omega^h = \omega^{lh}$, as defined in (21). Thus, the two firms play the same mass at their respective upper bounds.

We still need to check, as we had initially assumed, that $\underline{p}^H \geq \theta^L q^H$. This requires

$$\mu < \mu^+ \equiv \frac{\pi^H - \left(\theta^L q^H - c^H\right)}{\pi^H + \left(\theta^L q^H - c^H\right)}. \tag{22}$$

If $c^H \geq \theta^L q^H$ then $\mu^+ \geq 1$ so this condition is always satisfied. In contrast, if $c^H < \theta^L q^H$, then $\mu^+ < 1$. Hence, the equilibrium characterization above is only valid for $\mu \in (\widehat{\mu}, \mu^+)$.

We also need to make sure that firm H does not want to deviate outside of the support to serve both high as well as low types. The condition to guarantee this, which only applies for the case $c^H < \theta^L q^H$, is

$$\frac{1+\mu}{2}(1-\lambda)\left(\underline{p}^H - c^H\right) > \frac{1+\mu}{2}\left(\theta^L q^H - c^H\right)$$

given that the low types never buy good H unless it gives them positive utility. Rearranging this condition yields

$$\mu < \mu^- \equiv \frac{(1-\lambda)\pi^H - \left(\theta^L q^H - c^H\right)}{(1-\lambda)\pi^H + \left(\theta^L q^H - c^H\right)}. \tag{23}$$

It is easy to see that $\mu^- < \mu^+$ and from $(A3)$ that $\mu^- > \hat{\mu}$. Hence, for $\mu \in [\hat{\mu}, \mu^-)$ this equilibrium is guaranteed to exist (later on we will see that for $\mu > \mu^-$ another equilibrium is also guaranteed to exist, so equilibrium existence is not at stake).

Case 2: $\underline{p}^H \geq \theta^L q^H$ and the upper bound for firm H is $\theta^H q^H - q^L\Delta\theta$ Firm H cannot play a mass point at the upper bound $\theta^H q^H - q^L\Delta\theta$. If it did, firm LH could make more profits at prices slightly below the monopoly price than at the monopoly price.

Since the monopoly price must be in the support of firm LH, it follows that $\omega^h = 0$. Thus, firm LH's profits at the upper and bounds are

$$\begin{aligned}\Pi\left(LH, H; \bar{\mathbf{p}}\right) &= \frac{1+\mu}{2}\lambda\pi^L + \frac{1-\mu}{2}(1-\lambda)(\pi^H - \Delta\theta q^L) \\ \Pi\left(LH, H; \underline{\mathbf{p}}\right) &= \frac{1+\mu}{2}\lambda\pi^L + \frac{1+\mu}{2}(1-\lambda)(\underline{p}^H - c^H).\end{aligned}$$

Equalizing the two we obtain

$$\underline{p}^H = c^H + \frac{1-\mu}{1+\mu}(\pi^H - \Delta\theta q^L).$$

Since firm H must also be indifferent between playing $\underline{p}^H$ and $\theta^H q^H - \Delta\theta q^L - \epsilon$ (with $\epsilon \to 0$),

$$\left(\frac{1-\mu}{2} + \mu\omega^{lh}\right)(1-\lambda)(\pi^H - \Delta\theta q^L) = \frac{1-\mu}{2}(1-\lambda)(\pi^H - \Delta\theta q^L)$$

from which it follows that we should also have $\omega^{lh} = 0$. However, firm H would then rather deviate to charging the unconstrained monopoly price to obtain profits $\frac{1-\mu}{2}(1-\lambda)\pi^H$. It follows that this case can never be an equilibrium: if $\underline{p}^H > \theta^L q^H$, firm H must have the upper bound at the unconstrained monopoly price $\theta^H q^H$.

Cases 3 and 4: $\underline{p}^H < \theta^L q^H$ This case only arises when $c^H < \theta^L q^H$ as otherwise the firm would never price below $\theta^L q^H$. When this is case, we have to consider the possibility that $\left(IC^L\right)$ is binding. In other words, firm LH cannot always price good L at $\theta^L q^L$ given that for $p^H < \theta^L q^H$ the low types would rather buy good H. Hence, firm LH plays $p^L = \min\left\{\theta^L q^L, p^H - \theta^L \Delta q\right\}$ and both firms choose p^H randomly. Note that for $p^H \geq \theta^L q^H$ this results in $p^L = \theta^L q^L$ with per-unit profits on good L equal to π^L; whereas for $p^H < \theta^L q^H$ this results in $p^L = p^H - \theta^L \Delta q$ with per-unit profits on good L equal to $p^H - c^H + \varphi^L$.

We also first note that when playing $\underline{p}^H$, firm H attracts all the shoppers, both high and low types, with probability one. The reason is simple. First, whenever firm LH prices good H above $\theta^L q^H$, its price for L is $\theta^L q^L$ so the low types' utility from choosing L is zero and hence, they would rather buy H at $\underline{p}^H < \theta^L q^H$. And second, whenever firm LH prices good H below $\theta^L q^H$ but above $\underline{p}^H$, the low types are indifferent between choosing H or L from firm LH. Hence, when firm H prices at $\underline{p}^H$, its price for good H is lower than that of firm LH. It follows that the shopper low types prefer to buy good H from firm H rather than good L from firm LH.

We again need to consider two cases for firm H's upper bound.

Case 3: $\underline{p}^H < \theta^L q^H$ and the upper bound for firm H is $\theta^H q^H$ Firm H has to be indifferent between playing $\theta^H q^H$, which it plays with some positive probability ω^h, and the lower bound $\underline{p}^H$. So, it must hold that

$$\frac{1-\mu}{2}(1-\lambda)\pi^H = \left(\frac{1-\mu}{2}+\mu\right)\left(\underline{p}^H - c^H\right) \tag{24}$$

leading to

$$\underline{p}^H = c^H + \frac{1-\mu}{1+\mu}(1-\lambda)\pi^H.$$

We also need to show $\underline{p}^H < \theta^L q^H$ as assumed, which requires $\mu > \mu^-$, as defined above in equation (23).

On the other hand, if firm LH plays the upper bounds, $\overline{p}^L = \theta^L q^L$ and $\overline{p}^H = \theta^H q^H - \Delta\theta q^L$, it obtains

$$\Pi\left(LH, H; \bar{\mathbf{p}}\right) = \left[\frac{1-\mu}{2} + \mu(1 - F^h(\theta^L q^H))\right]\lambda\pi^L + \left[\frac{1-\mu}{2} + \mu\omega^h\right](1-\lambda)(\pi^H - \Delta\theta q^L) \tag{25}$$

where $F^h(\theta^L q^H)$ is the probability that firm H prices good H below $\theta^L q^H$. The profit from pricing at the upper bound must be equal to pricing good H at $\theta^L q^H$ and good L at $\theta^L q^L$, giving firm LH a profit of

$$\begin{aligned}\Pi\left(LH, H; \theta^L q^L, \theta^L q^H\right) &= \left[\frac{1-\mu}{2} + \mu(1 - F^h(\theta^L q^H))\right]\lambda\pi^L \\ &\quad + \left[\frac{1-\mu}{2} + \mu(1 - F^h(\theta^L q^H))\right](1-\lambda)(\theta^L q^H - c^H).\end{aligned} \tag{26}$$

Note that the probability that shoppers buy from firm LH, regardless of their type, is $1 - F^h(\theta^L q^H)$. If so, high types buy the high quality product, their preferred choice, while low types buy the low quality product, also their preferred choice. Since $\theta^L q^H - c^H = \pi^L - \varphi^L > 0$, expression (26) can be conveniently re-written as

$$\Pi\left(LH, H; \theta^L q^L, \theta^L q^H\right) = \left[\frac{1-\mu}{2} + \mu(1 - F^h(\theta^L q^H))\right]\left[\pi^L - (1-\lambda)\varphi^L\right]. \tag{27}$$

Either (25) or (27) must also be equal to what firm LH gets by playing the lower bounds $\underline{p}^H$ and $\underline{p}^L = \underline{p}^H - \theta^L \Delta q$, which is (note that the $\left(IC^L\right)$ is now binding at the lower bound)

$$\Pi\left(LH, H; \underline{\mathbf{p}}\right) = \frac{1+\mu}{2}\lambda\left(\underline{p}^H - c^H + \varphi^L\right) + \frac{1+\mu}{2}(1-\lambda)(\underline{p}^H - c^H). \tag{28}$$

Rearranging this latter expression and using (24) (note that at the lower bound firm LH makes more profits than firm H since it can discriminate), equation (28) reduces to

$$\Pi\left(LH, H; \underline{\mathbf{p}}\right) = \frac{1-\mu}{2}(1-\lambda)\pi^H + \frac{1+\mu}{2}\lambda\varphi^L. \tag{29}$$

Equalizing the above expression to (27) gives an implicit expression for $F^h(\theta^L q^H)$,

$$\frac{1-\mu}{2} + \mu(1 - F^h(\theta^L q^H)) = \frac{1}{2}\frac{(1-\mu)\,(1-\lambda)\pi^H + (1+\mu)\,\lambda\varphi^L}{\pi^L - (1-\lambda)\varphi^L} \equiv \Gamma. \tag{30}$$

Notice that $\lim_{\mu\to 1}\mu(1 - F^h(\theta^L q^H)) = \lambda\varphi^L/(\pi^L - (1-\lambda)\varphi^L) < 1$.

To conclude the characterization of the equilibrium, we need to find an expression for ω^h, the mass that firm H places at the upper bound, and make sure it is in the unit interval for all $\mu \in (\mu^-, 1)$. Using (30) while equalizing (25) and (29) yields

$$\mu\omega^h(1-\lambda)(\pi^H - \Delta\theta q^L) = \frac{1+\mu}{2}\lambda\varphi^L + \frac{1-\mu}{2}(1-\lambda)\Delta\theta q^L - \Gamma\lambda\pi^L.$$

Using $\pi^L - \varphi^L > 0$, it is easy to show that

$$\lim_{\mu\to 1}\omega^h = \frac{\lambda\varphi^L(\pi^L - \varphi^L)}{(\pi^L + \varphi^H)(\pi^L - (1-\lambda)\varphi^L)} \in (0,1).$$

It is also possible to show that $\omega^h < 1$ for $\mu = \mu^-$. At this point firm H's lower bound approaches $\theta^L q^H$, which is the deviation used in Case 1 to establish μ^-. Hence, there is continuity in the mixed strategy equilibria on either side of μ^-.

Finally, note we do not need to check firms' incentives to deviate outside of the support since at the lower bound firms are already serving all shoppers.

Case 4: $\underline{p}^H < \theta^L q^H$ and the upper bound for firm H is $\theta^H q^H - q^L\Delta\theta$ Firm H cannot play a mass point at the upper bound. If it did, firm LH could make more profits with a slight undercut of the monopoly price. Since the monopoly price must be in the support of firm LH, it follows that $\omega^h = 0$. Thus, equilibrium profits for the LH firm at the upper bound are

$$\Pi\,(LH, H; \bar{\mathbf{p}}) = \left(\frac{1-\mu}{2} + \mu\left(1 - F^h(\theta^L q^H)\right)\right)\lambda\pi^L + \frac{1-\mu}{2}(1-\lambda)(\pi^H - \Delta\theta q^L), \tag{31}$$

where $F^h(\theta^L q^H)$ is again the probability that firm H prices H good below $\theta^L q^H$. In equilibrium, these profits must be equal to those from pricing at the lower bound

$$\Pi\left(LH, H; \underline{\mathbf{p}}\right) = \frac{1+\mu}{2}\lambda(\underline{p}^H - \theta^L\Delta q - c^L) + \frac{1+\mu}{2}(1-\lambda)(\underline{p}^H - c^H).$$

In addition, we know that firm H must be indifferent between playing $\theta^H q^H - \Delta\theta q^L - \epsilon$ (with $\epsilon \to 0$) and $\underline{p}^H$, implying

$$\left(\frac{1-\mu}{2} + \mu\omega^{lh}\right)(1-\lambda)(\pi^H - \Delta\theta q^L) = \left(\frac{1-\mu}{2} + \mu\right)(\underline{p}^H - c^H).$$

This gives us two equations for three unknowns, namely, $F^h(\theta^L q^H)$, ω^{lh}, and $\underline{p}^H$. A third equation is obtained from LH's indifference between pricing good H at $p^H = \theta^L q^H$ (and L at $\theta^L q^L$) and at any higher price in the support (and L still at $\theta^L q^L$). At the upper bound, this indifference reduces to

$$\left[\frac{1-\mu}{2} + \mu\left(1 - F^h(\theta^L q^H)\right)\right](1-\lambda)(\theta^L q^H - c^H) = \frac{1-\mu}{2}(1-\lambda)(\pi^H - \Delta\theta q^L),$$

from where we obtain

$$\mu\left(1 - F^h(\theta^L q^H)\right) = \frac{1-\mu}{2}\frac{\Delta\theta\Delta q}{\theta^L q^H - c^H}.$$

Replacing it into (31) yields firm LH's equilibrium profits,

$$\Pi\left(LH, H\right) = \frac{1-\mu}{2}\left[\left(1 + \frac{\Delta\theta\Delta q}{\theta^L q^H - c^H}\right)\lambda\pi^L + (1-\lambda)(\pi^H - \Delta\theta q^L)\right].$$

Since in equilibrium $1 - F^h(\theta^L q^H) \leq 1$, which implies that $\mu \geq \Delta\theta\Delta q/[2(\theta^L q^H - c^H) + \Delta\theta\Delta q]$, it is evident that the above characterization cannot be an equilibrium, since as $\mu \to 1$ firm LH would be making less than its minmax profit of $\lambda\varphi^L$.

Equilibrium profits under (H, LH) Wrapping up, equilibrium pricing and equilibrium profits are characterized as follows:

If $\mu \leq \widehat{\mu}$: PSE with firm H charging the unconstrained monopoly prices, and firm LH charging the constrained ones. Equilibrium profits are those in equations (16) and (17).

If $\mu \in (\widehat{\mu}, 1)$: MSE with firm H charging the unconstrained monopoly price at the upper bound. Equilibrium profits are

$$\Pi\left(H, LH\right) = \frac{1-\mu}{2}(1-\lambda)\pi^H.$$

For firm LH, equilibrium profits depend on μ: If $c^H \geq \theta^L q^H$ and $\mu \in (\widehat{\mu}, 1)$, or if $c^H < \theta^L q^H$ and $\mu \in (\widehat{\mu}, \mu^-]$,

$$\Pi\left(LH, H\right) = \frac{1+\mu}{2}\lambda\pi^L + \frac{1-\mu}{2}(1-\lambda)\pi^H.$$

Otherwise,

$$\Pi\left(LH, H\right) = \frac{1+\mu}{2}\lambda\left(\Delta c - \theta^L \Delta q\right) + \frac{1-\mu}{2}(1-\lambda)\pi^H. \tag{32}$$

Q.E.D.

Proof of Proposition 6 [pricing at subgame (L, H)] We start by showing that for low values of μ there exists a PSE. The natural candidate is $p^H = \theta^H q^H$ for firm H and $p^L = \theta^L q^L$ for firm L. Firm H only sells to the non-shopper high types, while firm L sells to all the rest (except for the non-shoppers that visit firm H). Thus, firms' profits are

$$\begin{aligned} \Pi(H, L) &= \frac{1-\mu}{2}(1-\lambda)\pi^H \\ \Pi(L, H) &= \frac{1+\mu}{2}\pi^L. \end{aligned} \tag{33}$$

For this to be an equilibrium, it must be the case that neither firm wants to deviate from it. In particular, firm H could charge $\theta^H q^H - q^L \Delta\theta$ to also attract the shopper high types. It would thus sell to all the high types (except for the non-shoppers who visit firm L) and would make profits

$$\Pi'(H, L) = \frac{1+\mu}{2}(1-\lambda)(\pi^H - q^L \Delta\theta). \tag{34}$$

Comparing the profit expressions for firm H, the PSE equilibrium exists if and only if $\mu \leq \widehat{\mu}$ as defined in (19). Otherwise, the equilibrium must be in mixed strategies. Note we do not need to check that firm does not want to charge $\theta^L q^H - c^H$ so as to serve all the non-shoppers, including the low types, given that $(A3)$ guarantees that $(1-\lambda)\pi^H > \theta^L q^H - c^H$.

In order to characterize the MSE, let us start by discussing the properties of the price supports. The upper bound for firm L must be $\theta^L q^L$, as for the low quality product the constrained and unconstrained monopoly prices coincide. Following the PSE discussion above, the upper bound for firm H could either be the unconstrained monopoly price $\theta^H q^H$ or the constrained monopoly price $\theta^H q^H - q^L \Delta\theta$. In either case, firm H never plays a price in between these two prices. Before considering these two possibilities, we first note that firm L has to play a mass at its upper bound $\theta^L q^L$. Otherwise, when firm H charged $\theta^H q^H - q^L \Delta\theta$, since firm L would be charging prices below $\theta^L q^L$ with probability one, firm H would make strictly lower profits than if it charged $\theta^H q^H$, as it would serve the same set of consumers at a lower price. Accordingly, firm L must play a mass point at $\overline{p}^L = \theta^L q^L$, which we denote by $\omega^l > 0$. Similarly, let ω^h denote the mass point that firm H puts on its upper bound. As for the lower bounds, we only need to consider case $\underline{p}^H \geq \theta^L q^H$ because $\underline{p}^H < \theta^L q^H$ is ruled out by $(A3)$.

Case 1: $\underline{p}^H > \theta^L q^H$ and the upper bound for firm H is $\theta^H q^H$ For given $\underline{p}^L$, it never pays firm H to charge less than $\underline{p}^L + \theta^H \Delta q$ since at this price it attracts all the shopper high types. Hence, we must have $\underline{p}^H \geq \underline{p}^L + \theta^H \Delta q$. Second, for given $\underline{p}^H$, it never pays firm L to charge less than $\underline{p}^H - \theta^H \Delta q$ since at this price it attracts all the shoppers. Hence, we must have $\underline{p}^L \geq \underline{p}^H - \theta^H \Delta q$. Putting these two conditions together, it follows that we must have $\underline{p}^H = \underline{p}^L + \theta^H \Delta q$. Therefore, when firm H charges $\underline{p}^H$, with probability one firm L is charging prices above $\underline{p}^H - \theta^H \Delta q$. Hence, all the high types buy from firm H (except for the non-shoppers that visit L) implying that firm H's profits in equilibrium must satisfy

$$\begin{aligned} \Pi\left(H, L; \underline{p}^H\right) &= \frac{1+\mu}{2}(1-\lambda)\left(\underline{p}^H - c^H\right) \\ &= \frac{1+\mu}{2}(1-\lambda)\left(\underline{p}^L - c^L + \varphi^H\right). \end{aligned} \tag{35}$$

Similarly, when firm L charges $\underline{p}^L$, with probability one firm H is charging prices above $\underline{p}^L + \theta^H \Delta q$. Hence, the firm serves its non-shoppers plus all the shoppers, both high and low types, implying the firm L's profits in equilibrium must satisfy

$$\Pi(L, H) = \frac{1+\mu}{2}\left(\underline{p}^L - c^L\right). \tag{36}$$

Since firm H's upper bound is $\theta^H q^H$ and firm L is pricing below $\theta^L q^L$ with probability one, firm H's profits are

$$\Pi(H, L) = \frac{1-\mu}{2}(1-\lambda)\pi^H. \tag{37}$$

Since firm H must be indifferent between playing $\theta^H q^H$ and $\theta^H q^H - \Delta\theta q^L$, it follows that the mass that firm L puts on the monopoly price in case 1, ω_1^l, must satisfy

$$\left(\frac{1-\mu}{2} + \mu\omega_1^l\right)(1-\lambda)\left(\pi^H - q^L \Delta\theta\right) = \frac{1-\mu}{2}(1-\lambda)\pi^H$$

which is satisfied at ω_1^l equal to expression (21). Just as we argued then, there is continuity between the pure strategy and this mixed-strategy equilibrium.

Firm H must also be indifferent between playing the upper bound $\theta^H q^H$ and $\underline{p}^H$. Equating profits yields $\underline{p}^H$ as in equation (20).[48] Since we must have $\underline{p}^H = \underline{p}^L + \theta^H \Delta q$, then

$$\underline{p}^L = c^L + \frac{1-\mu}{1+\mu}\pi^H - \varphi^H.$$

Using equation (36), equilibrium profits for firm L are thus

$$\Pi(L, H) = \frac{1+\mu}{2}\left(\frac{1-\mu}{1+\mu}\pi^H - \varphi^H\right).$$

When firm L prices at the upper bound $\theta^L q^L$, it makes profits

$$\Pi\left(L, H; \overline{p}^L\right) = \left[\frac{1+\mu}{2}\lambda + \left(\frac{1-\mu}{2} + \mu\omega^h\right)(1-\lambda)\right]\pi^L. \tag{38}$$

[48] Note that as $\mu \to 1$ the firm would be making zero profits, which cannot be the case since that is below its minmax. Hence, at some point his equilibrium must cease to exist, as we will see below.

Equalizing these two expressions and rearranging terms yields

$$\omega^h = \frac{1-\mu}{2}\frac{\left(\pi^H - \pi^L - \varphi^H\right)}{\mu\left(1-\lambda\right)\pi^L} - \frac{\lambda\pi^L + \varphi^H}{\left(1-\lambda\right)\pi^L}. \tag{39}$$

Note that for $\mu \to \widehat{\mu}$, $\omega^h \to 1$. Hence, there is continuity between the pure strategy equilibrium and this mixed-strategy equilibrium. For

$$\mu = \widetilde{\mu} \equiv \frac{\pi^H - \left(\pi^L + \varphi^H\right)}{\pi^H + \left(\pi^L + \varphi^H\right) - 2\left(1-\lambda\right)\pi^L} \tag{40}$$

we have $\omega^h = 0$. Since ω^h cannot become negative, this equilibrium cannot exist for $\mu > \widetilde{\mu}$ (below we elaborate further on this).

Last, we need to check, as we have assumed, that $\underline{p}^H > \theta^L q^H$. This requires $\mu < \mu^+$ as defined in (22). If $c^H \geq \theta^L q^H$ then $\mu^+ \geq 1$ so this condition is always satisfied. In contrast, if $c^H < \theta^L q^H$, then $\mu^+ < 1$. Hence, the equilibrium characterization above is only valid for $\mu < \mu^+$ and $\mu < \widetilde{\mu}$. Using the fact that $\pi^L + \varphi^H > \theta^L q^H - c^H = \pi^L - \varphi^L$ and $\pi^H > \theta^L q^H - c^H$, it can be established that $\widetilde{\mu} < \mu^+$. Therefore, the equilibrium characterized here only applies for $\mu \in (\widehat{\mu}, \widetilde{\mu}]$.

We still need to check that firm H does not want to deviate outside of the support for this characterization to be valid. If firm H deviates to serve all the low type shoppers with probability one by charging $p^H = \underline{p}^L + \theta^L \Delta q$, it would get a profit of $\frac{1+\mu}{2}\left(\underline{p}^L + \theta^L \Delta q - c^H\right)$. For this latter to be strictly lower than $\frac{1+\mu}{2}(1-\lambda)\left(\underline{p}^H - c^H\right)$ we need

$$\varphi^L + \varphi^H > \lambda \frac{1-\mu}{1+\mu}\pi^H \tag{41}$$

to hold for all $\mu \in (\widehat{\mu}, \widetilde{\mu}]$. So it suffices to show that (41) holds for $\widehat{\mu}$, which it strictly does because of $(A3)$.

To continue with the equilibrium characterization we need to consider now the possibility of a smaller upper bound for firm H.

Case 2: $\underline{p}^H > \theta^L q^H$ and the upper bound for firm H is $\theta^H q^H - q^L \Delta\theta$ Firm H cannot play a mass point at the upper bound $\theta^H q^H - q^L \Delta\theta$. If it did, firm L could make more profits at prices slightly below the monopoly price than at the monopoly price. Since the monopoly price must be in the support of firm L, it follows that $\omega^h = 0$. Thus, equilibrium profits for the L firm at the upper and lower bounds are

$$\begin{aligned}\Pi\left(L, H; \overline{p}^L\right) &= \left(\frac{1-\mu}{2} + \mu\lambda\right)\pi^L \\ \Pi\left(L, H; \underline{p}^L\right) &= \frac{1+\mu}{2}\left(\underline{p}^L - c^L\right).\end{aligned}$$

Equalizing the two yields

$$\underline{p}^L - c^L = \left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right)\pi^L.$$

So that

$$\underline{p}^H - c^H = \left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right)\pi^L + \left(\theta^H \Delta q - \Delta c\right). \tag{42}$$

Last, we again need to check that $\underline{p}^H > \theta^L q^H$ as initially assumed, and this requires

$$\underline{p}^H = c^H + \left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right)\pi^L + \varphi^H > \theta^L q^H \tag{43}$$

$$\mu < \frac{(1-\lambda)\pi^L - \theta^L q^H + c^H + \varphi^H + \lambda\pi^L}{(1-\lambda)\pi^L + \theta^L q^H - c^H - \varphi^H - \lambda\pi^L} \equiv \breve{\mu}$$

Note that $\breve{\mu} \geq 1$ iff

$$\pi^L - \varphi^L \leq \varphi^H + \lambda\pi^L \tag{44}$$

which always holds by assumption ($A2$). Hence, $\underline{p}^H \geq \theta^L q^H$ is always satisfied. In turn, this also implies that we do not need to consider the cases with $\underline{p}^H < \theta^L q^H$.

Using equation (35) to obtain firm H's equilibrium profits,

$$\Pi(H, L) = \frac{1+\mu}{2}(1-\lambda)\left[\left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right)\pi^L + \varphi^H\right]. \tag{45}$$

Since firm H must be indifferent between playing $\theta^H q^H - \Delta\theta q^L$ and $\underline{p}^H$, it follows that the mass that firm L puts at the upper bound in case 2, ω_2^l, must satisfy

$$\left(\frac{1-\mu}{2} + \mu\omega_2^l\right)(1-\lambda)\left(\pi^L + \varphi^H\right) = \frac{1+\mu}{2}(1-\lambda)\left(\frac{1-\mu+2\lambda\mu}{1+\mu}\pi^L + \varphi^H\right).$$

which, after some algebra, yields

$$\omega_2^l = \frac{\varphi^H + \lambda\pi^L}{\varphi^H + \pi^L}. \tag{46}$$

The expressions for ω_1^l and ω_2^l cross at a single value of μ, call it μ^+. Since ω_1^l is decreasing in μ and ω_2^l is flat, $\omega_1^l \geq \omega_2^l$ if and only if $\mu \leq \mu^+$. At μ^+, since $\omega_1^l = \omega_2^l$, the profits made by firm H in cases 1 and 2 must coincide, i.e., expressions (37) and (45) must be equal. At μ^+, using expressions (37) and (45),

$$\frac{1-\mu}{2}(1-\lambda)\pi^H = \frac{1+\mu}{2}(1-\lambda)\left(\frac{1-\mu+2\lambda\mu}{1+\mu}\pi^L + \varphi^H\right)$$

and rearranging,

$$\frac{1+\mu}{2}\left(\frac{1-\mu}{1+\mu}\pi^H - \varphi^H\right) = \left(\frac{1+\mu}{2}\lambda + \left(\frac{1-\mu}{2}\right)(1-\lambda)\right)\pi^L.$$

This equation is satisfied at $\mu = \widetilde{\mu}$ (from equation (39), recall how we defined $\widetilde{\mu}$). Hence, we must have $\mu^+ = \widetilde{\mu}$.

We still need to check for case 2 that firm H does not want to deviate outside of the support. Following the analysis of case 1 and noticing that now $\underline{p}^H - c^H$ is given by (42), the condition for firm H not to find it profitable to deviate to serve all the shopper low types with probability one by charging $p^H = \underline{p}^L + \theta^L \Delta q$ reduces to

$$\Delta c - \theta^L \Delta q + (1-\lambda)\varphi^H > \lambda \left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda) \right) \pi^L$$

for all $\mu \geq \widetilde{\mu}$. Since the term in parenthesis in the RHS is less than 1, we just need $\Delta c - \theta^L \Delta q + (1-\lambda)\varphi^H > \lambda \pi^L$, which rearranged is simply $(A3)$.

The above characterization has important implications for equilibrium existence and uniqueness. First, for $\mu < \widetilde{\mu}$, $\omega_1^l > \omega_2^l$ implies that firm H makes more profits at the unconstrained monopoly price than at the constrained monopoly price. Hence, the equilibrium characterized in case 2 cannot exist: the unique equilibrium is the one characterized in case 1. Second, for $\mu \geq \widetilde{\mu}$, profits for firm L in case 1 fall below its minmax, so that the unique equilibrium is the one characterized in case 2. Last note that the equilibrium is continuous in μ. In particular, the masses with which firms play their upper bounds are continuous in μ at $\widehat{\mu}$ as we move form the PSE to the MSE in case 1, and at $\widetilde{\mu}$ when we move to the MSE in cases 1 to 2.

Equilibrium profits under (H, L) Depending on the value of μ, the equilibrium is characterized as follows:

For $\mu \in [0, \widehat{\mu}]$:

$$\begin{aligned} \Pi(H,L) &= \frac{1-\mu}{2}(1-\lambda)\pi^H \\ \Pi(L,H) &= \frac{1+\mu}{2}\pi^L. \end{aligned} \tag{47}$$

For $\mu \in [\widehat{\mu}, \widetilde{\mu}]$:

$$\begin{aligned} \Pi(H,L) &= \frac{1-\mu}{2}(1-\lambda)\pi^H \\ \Pi(L,H) &= \frac{1+\mu}{2}\left(\frac{1-\mu}{1+\mu}\pi^H - \varphi^H\right). \end{aligned} \tag{48}$$

For $\mu \in [\widetilde{\mu}, 1]$:

$$\begin{aligned} \Pi(H,L) &= \frac{1+\mu}{2}(1-\lambda)\left[\left(\lambda + \frac{1-\mu}{1+\mu}(1-\lambda)\right)\pi^L + \varphi^H\right] \\ \Pi(L,H) &= \left[\frac{1+\mu}{2}\lambda + \frac{1-\mu}{2}(1-\lambda)\right]\pi^L. \end{aligned} \tag{49}$$

where $\widehat{\mu}$ and $\widetilde{\mu}$ are defined, respectively, by (19) and (40). *Q.E.D.*

Proof of Proposition 7 [quality choices] Each firm has four potential choices: $\{\varnothing, L, H, LH\}$. On the one hand, to prove that (LH, LH) is a SPE of the game for all $\mu < 1$, just note that all payoffs $\Pi(LH, LH)$, $\Pi(H, LH)$ and $\Pi(L, LH)$ are proportional to $(1-\mu)/2$ so that $(A1)$ allows to conclude that $\Pi(LH, LH)$ is the greatest among these, just as in the monopoly case.

On the other hand, to find the conditions under which (L, H) is an equilibrium, we need to assess firm L's deviation gains when also carrying good H (it is easy to check that this is the critical deviation; for instance, let $\mu \to 1$ and use $(A2)$ to note that firm L's deviation gains are greater than firm H's, i.e., $\Pi(LH, H) - \Pi(L, H) > \Pi(LH, L) - \Pi(H, L)$). Since the relevant payoffs are (49) and (32), this gain is equal to

$$\Pi(L, H) - \Pi(LH, H) = \frac{1+\mu}{2}\lambda\left(\pi^L - \varphi^L\right) - \frac{1-\mu}{2}(1-\lambda)\left(\pi^H - \pi^L\right). \tag{50a}$$

Solving for μ, the above profit difference is positive iff $\mu \geq \mu^*$, where

$$\mu^* = \frac{(1-\lambda)\left(\pi^H - \pi^L\right) - \lambda\left(\pi^L - \varphi^L\right)}{(1-\lambda)\left(\pi^H - \pi^L\right) + \lambda\left(\pi^L - \varphi^L\right)}.$$

Note that when $c^H \geq \theta^L q^H$, $\pi^L = \varphi^L$ and $\mu^* = 1$, making $c^H \geq \theta^L q^H$ a sufficient condition for the uniqueness of (LH, LH). Furthermore, taking the derivative of μ^* with respect to c^H shows that

$$\frac{\partial \mu^*}{\partial c^H} = -2\lambda(1-\lambda)\frac{\left(\pi^H - \pi^L\right) - \left(\pi^L - \varphi^L\right)}{\left((1-\lambda)\left(\pi^H - \pi^L\right) + \lambda\left(\pi^L - \varphi^L\right)\right)^2}.$$

So that

$$\begin{aligned} sign\left\{\frac{\partial \mu^*}{\partial c^H}\right\} &= -sign\left\{\left(\pi^H - \pi^L\right) - \left(\pi^L - \varphi^L\right)\right\} \\ &= -sign\left\{c^L - \theta^L q^H\right\} > 0. \end{aligned}$$

Last, there might also exist a symmetric MSE such that firms choose L and H randomly, just as we showed in the proof of Proposition 2. This equilibrium constitutes a SPE if and only if it is not dominated to choosing LH, i.e.,

$$\Pi(L, H)\left[\Pi(L, L) - \Pi(LH, L)\right] + \Pi(H, L)\left[\Pi(L, H) - \Pi(LH, H)\right] \geq 0$$

or equivalently, iff

$$\Pi(L, H) - \Pi(LH, H) \geq \frac{\Pi(L, H)}{\Pi(H, L)}\left[\Pi(LH, L) - \Pi(L, L)\right] > 0.$$

Hence, whereas the existence of the asymmetric PSE (L, H) requires the profit difference (50a) to be positive, the existence of the MSE requires such a difference to be greater than a strictly positive number. If we denote with μ^{**} the critical value for the existence of the MSE, we must then have $\mu^{**} \geq \mu^{*}$. It thus follows that for $\mu < \mu^{*}$, the unique equilibrium (either pure or mixed) is (LH, LH). *Q.E.D.*

Proof of Lemma 3 [prices and consumers surplus at the SPE] (i) It is straightforward to see that, conditional on firms playing (LH, LH), expected prices are decreasing in μ. Since there is full discrimination, total consumption of each good remains fixed so that total surplus is given by $\lambda\pi^L + (1-\lambda)\pi^H$, irrespectively of μ. Since profits in equation (3) decrease in μ, consumer surplus must increase in μ. In turn, this implies that expected prices must be decreasing in μ. For given parameter values, competition is stronger at subgame (LH, LH) than at (L, H). Hence, expected prices at the former must be lower and consumer surplus must be higher. Thus, as μ goes down, expected prices (consumer surplus) at (LH, LH) decrease continuously until they jump up when firms start playing (L, H), either at $\mu \to 1$ or at $\mu \to \mu^{*}$ depending on equilibrium selection. Similarly, as μ goes down, consumer surplus at (LH, LH) increases continuously until it jumps down when firms start playing (L, H), either at $\mu \to 1$ or at $\mu \to \mu^{*}$ depending on equilibrium selection.

(ii) Last, expected prices at (LH, LH) are equal to the (constrained) monopoly prices for $\mu = 0$ and to marginal costs for $\mu \to 1$. By Proposition 1, expected prices at (L, H) for $\mu = 1$ are strictly above marginal costs. It follows that there must exist $\mu' \in (0, 1)$ such that expected prices at (LH, LH) for $\mu \in (\mu', 1)$ are lower than at (L, H) for $\mu = 1$. If $\mu' < \mu^{*}$, it follows that regardless of equilibrium selection, expected prices are lower at $\mu \in (\mu', \mu^{*})$ than at $\mu = 1$. Similarly, for $\mu = 0$, expected consumer surplus at (LH, LH) is equal to the expected information rents of the high types, $\Delta\theta q^L$, for $\mu = 0$, and it is equal to total welfare for $\mu \to 1$ as in this case profits are zero. Total welfare is the same at (LH, LH) for all μ than at (L, H) at $\mu = 1$ since in both cases discrimination is complete. The proof is completed by noting that, by (i) above, consumer surplus at (LH, LH) is increasing in μ. *Q.E.D.*

References

[1] Armstrong, M. (2015), Search and Rip-offs Externalities, *Review of Industrial Organization* 47(3), 273–302.

[2] Armstrong, M. (2016), Ordered Consumer Search, CEPR Discussion Papers 11566.

[3] Anderson, S. and Renault, R. (1999), Pricing, Product Diversity, and Search Costs: A Bertrand-Chamberlin-Diamond Model, *RAND Journal of Economics* 30(4), 719-735.

[4] Baye, M. and J. Morgan (2001), Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets, *American Economic Review* 91(3), 454-474.

[5] Borenstein, S. and Rose, N.L. (1994), Competition and Price Dispersion in the U.S. Airline Industry, *Journal of Political Economy*, 102, 653-68.

[6] Brander, J. and Eaton, J. (1984), Product-line Rivalry, *American Economic Review* 74, 323-334.

[7] Burdett, K. and Judd, K.L. (1983), Equilibrium Price Dispersion, *Econometrica* 51(4), 955-969.

[8] Busse, M. and Rysman, M. (2005), Competition and Price Discrimination in Yellow Pages Advertising, *RAND Journal of Economics* 36(2), 378-390.

[9] Bar-Isaac, H., Caruana, G. and Cuñat, V. (2012), Search, Design, and Market Structure, *American Economic Review* 102, 1140-60.

[10] Chamberlin, E. H. (1933), The Theory of Monopolistic Competition. Cambridge, MA: Harvard University Press.

[11] Champsaur, P. and Rochet, J.-C. (1989), multi-product Duopolists, *Econometrica* 57, 533-557.

[12] Chen, L., Mislove, A. and Wilson, C. (2016), An Empirical Analysis of Algorithmic Pricing on Amazon Marketplace, mimeo.

[13] Crawford, G. and Shum, M. (2007), Monopoly Quality Degradation in Cable Television, *Journal of Law and Economics* 50.

[14] Diamond, P. (1971), A Model of Price Adjustment, *Journal of Economic Theory* 3, 156-168.

[15] De Los Santos, B., Hortaçsu, A. and Wildenbeest, M. (2012), Testing Models of Consumer Search Using Data on Web Browsing and Purchasing Behavior, *American Economic Review* 102(6), 2955-80.

[16] Ellison, G. (2005), A Model of Add-on Pricing, *Quarterly Journal of Economics* 120(2), 585-637.

[17] Ershow, D. (2016), The Effect of Consumer Search Costs on Entry and Quality in the Mobile App Market, mimeo, University of Toronto.

[18] Fabra, N. and Montero, J.-P. (2017), Product Choice and Price Discrimination in Markets with Search Costs, CEPR Discussion Paper 12046.

[19] Fabra, N. and Reguant, M. (2017), Do Large Buyers Pay Lower Prices? The Role of Search Costs, mimeo, Northwestern University.

[20] Gal-Or, E. (1983), Quality and Quantity Competition, *Bell Journal of Economics* 14, 590-600.

[21] Gamp, (2017), Guided Search, mimeo.

[22] Garret, D., Gomes, R., and Maestri, L. (2016), Competitive Screening under Heterogeneous Information, mimeo, Toulouse School of Economics.

[23] Gerardi, K. and Shapiro, A. (2009), Does Competition Reduce Price Dispersion? New Evidence from the Airline Industry, *Journal of Political Economy* 117, 1-37.

[24] Gilbert, R. and Matutes, C. (1993), Product Line Rivalry with Brand Differentiation, The Journal of Industrial Economics 41(3), 223-240.

[25] Hong, H., and Shum, M. (2006), Using Price Distributions to Estimate Search Costs, *RAND Journal of Economics* 37(2), 257-75.

[26] Iyer, G., and Seetharaman, P.B. (2003), To Price Discriminate or Not: Product Choice and the Selection Bias Problem, *Quantitative Marketing and Economics* 1, 155-178.

[27] Johnson, J.P., and Myatt, D.P. (2003), multi-product Quality Competition: Fighting Brands and Product Line Pruning, *American Economic Review*, 749-774.

[28] Johnson, J.P., and Myatt, D.P. (2006), multi-product Cournot Oligopoly, *RAND Journal of Economics* 37, 583-601.

[29] Johnson, J.P., and Myatt, D.P. (2015), The Properties of Product Line Prices, *International Journal of Industrial Organization* 43, 182-188.

[30] Kaplan, G., Menzio, G., Rudanko, L. and Thachter, N. (2016), Relative Price Dispersion: Evidence and Theory, NBER Working Paper No. 21931.

[31] Klemperer, P. (1992), Equilibrium Product Lines: Competing Head-to-Head May be Less Competitive, *American Economic Review* 82(4), 740-55.

[32] Leslie, P. (2004), Price Discrimination in Broadway Theater, *RAND Journal of Economics* 35, 520-541.

[33] Mc Manus, B. (2007), Nonlinear Pricing in an Oligopoly Market: The Case of Specialty Coffee, *RAND Journal of Economics* 38, 513-533.

[34] Moraga González, J.L., Sandor, Z. and Wildenbeest, M. (2017), Prices and Heterogeneous Search Costs, *RAND Journal of Economics* 48(1), 125-146.

[35] Mussa, M. and Rosen, S. (1978), Monopoly and Product Quality, *Journal of Economic Theory* 18, 301-317.

[36] Nevo, A. and Wolfram, C. (2002), Why Do Manufacturers Issue Coupons? An Empirical Analysis of Breakfast Cereals?, *RAND Journal of Economics* 33, 319-339.

[37] Petrikaite, V. (2017), Consumer Obfuscation by a multi-product Firm, mimeo.

[38] Rhodes, A. (2015), multi-product Retailing, *Review of Economic Studies* 82(1), 360-390.

[39] Rhodes, A., and Zhou, J. (2016), Consumer Search and Retail Market Structure, mimeo, Toulouse School of Economics.

[40] Romero, J. (2015), The Effect of Hysteresis on Equilibrium Selection in Coordination Games, *Journal of Economic Behavior and Organization* 111, 88-105.

[41] Seim, K. and Sinkinson, M. (2016), Mixed Pricing in Online Marketplaces, *Quantitative Marketing and Economics* 14(2), 129-155.

[42] Shaked, A. and Sutton, J. (1982), Relaxing Price Competition Through Product Differentiation, *The Review of Economic Studies* 49(1), 3-13.

[43] Shepard, A. (1991), Price Discrimination and Retail Configuration, *Journal of Political Economy* 99(1), 30-53.

[44] Stahl, D.O. (1989), Oligopolistic Pricing with Sequential Consumer Search, *American Economic Review* 79(4), 700-712.

[45] Stole, L. (1995), Nonlinear Pricing and Oligopoly, *Journal of Economics & Management Strategy* 4, 529-562.

[46] Stole, L. (2007), Price Discrimination and Competition, *Handbook of Industrial Organization*, 3 (eds. M. Armstrong and R. Porter), North-Holland.

[47] Sullivan, C. (2016), The Ice Cream Split: Empirically Distinguishing Price and Product Space Collusion, mimeo, University of Michigan.

[48] Tirole, J. (1988), The Theory of Industrial Organization, Cambridge, MA: MIT Press.

[49] Verboven, F. (1999), Product Line Rivalry and Market Segmentation - with an Application to Automobile Optional Engine Pricing, *The Journal of Industrial Economics* 47(4), 399-425.

[50] Villas-Boas, M. (2004), Communication Strategies and Product Line Design, *Marketing Science* 23, 304-316.

[51] Varian, H. (1980), A Model of Sales, *American Economic Review* 70(4), 651-659.

[52] Wernerfelt, B. (1986), Product Line Rivalry: Note, *American Economic Review* 76(4), 842-44.

[53] Wernerfelt, B. (1988), Umbrella Branding as a Signal of New Product Quality: An Example of Signalling by Posting a Bond, *RAND Journal of Economics* 19, 458-466.

[54] Wildenbeest, M. R. (2011), An Empirical Model of Search with Vertically Differentiated Products, *RAND Journal of Economics* 42(4), 729-757.

[55] Zhou, J. (2014), Multi-product Search and the Joint Search Effect, *American Economic Review*, 104(9), 2918-2939.